Geometric Properties of Toric Patches

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Second International Workshop on Algebraic Geometry and Approximation Theory

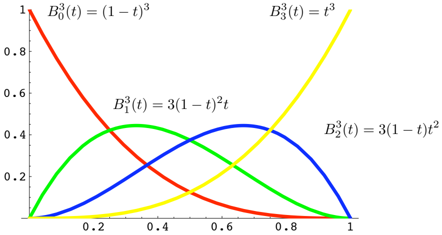
Outline

- Parametric patches
- Toric patches
- Properties:
- injectivity of toric patches
- toric deformations

Bézier curves

Bernstein polynomials

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}$$



Parametric definition

$$F(t) := \sum_{i=0}^{n} B_i^n(t)\mathbf{b}_i, \quad t \in [0,1]$$

where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ are (control) points in some affine space.

Properties of Bézier curves

Linear precision

$$\sum_{i=0}^{n} B_i^n(t)\frac{i}{n} = t$$

- $\mathcal{A} = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$.
- $\Delta = [0, 1]$ is the convex hull of $\mathcal{A}$.
- The Bernstein polynomials are indexed by $\mathcal{A}$ and have domain Δ.
- Linear precision means $F(t)$ is the identity on Δ when $\mathbf{b}_i = \frac{i}{n}$.

(Control point) patch schemes

Let $\mathcal{A} \subset \mathbb{R}^d$ (e.g. $d = 2$) be a finite index set with convex hull Δ.

$\beta := \{\beta_{\mathbf{a}} : \Delta \to \mathbb{R}_{\geq 0} \mid \mathbf{a} \in \mathcal{A}\}$, basis functions with $1 = \sum_{\mathbf{a}} \beta_{\mathbf{a}}(x)$.

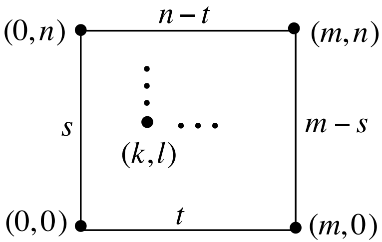
Given control points $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ (e.g. $\ell = 3$), get a map

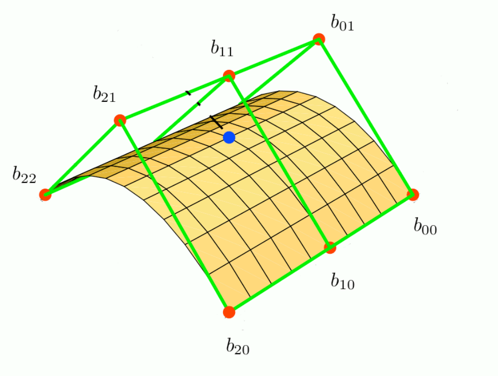
$$F : \Delta \longrightarrow \mathbb{R}^\ell \qquad x \longmapsto \sum \beta_{\mathbf{a}}(x)\mathbf{b}_{\mathbf{a}}$$

The image of F is a patch with shape Δ. Call $(\beta, \mathcal{A})$ a patch scheme.

Here, weights have been absorbed into the basis functions.

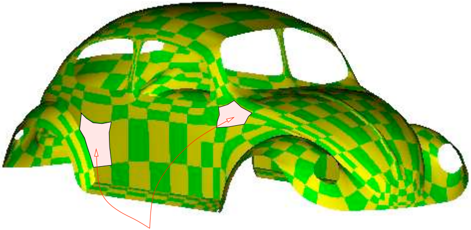
Bézier rectangles

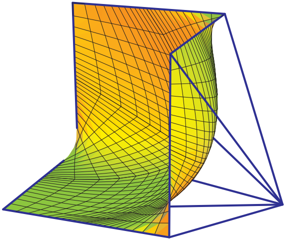




$$F(s,t) = \sum_{kl} \frac{\binom{m}{k}\binom{n}{l} s^k (m-s)^{m-k} t^l (n-t)^{n-l}}{m^m n^n} \mathbf{b}_{kl}$$

Multi-sided patches





Toric patches (after Krasauskas)

A polytope Δ with integer vertices is given by facet inequalities

$$\Delta = \left\{ x \in \mathbb{R}^d \mid h_i(x) \geq 0,\ i = 1, \ldots, r \right\},$$

where $h_i(x) = \mathbf{v}_i \cdot x + c_i$ with inward pointing primitive normal vector $\mathbf{v}_i$.

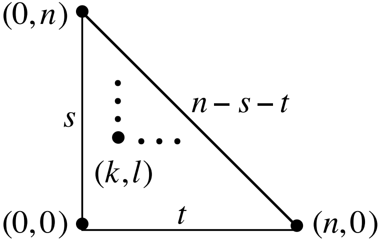
For each $\mathbf{a} \in \mathcal{A} := \Delta \cap \mathbb{Z}^d$, there is a toric Bézier function

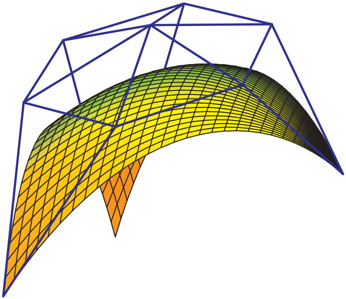
$$\beta_{\mathbf{a}}(x) := h_1(x)^{h_1(\mathbf{a})} h_2(x)^{h_2(\mathbf{a})} \cdots h_r(x)^{h_r(\mathbf{a})}.$$

Let $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}_{>}$ be positive weights. The toric patch $(w, \mathcal{A})$ is the patch with basis functions

$$\frac{w_{\mathbf{a}} \beta_{\mathbf{a}}}{\sum_{\mathbf{a} \in \mathcal{A}} w_{\mathbf{a}} \beta_{\mathbf{a}}} \qquad \text{for each } \mathbf{a} \in \mathcal{A}.$$

Bézier triangles





$$F(s,t) = \sum_{kl} \frac{\binom{n}{kl} s^k t^l (n-s-t)^{n-k-l}}{n^n} \mathbf{b}_{kl}$$

Toric varieties

Tautological map

Given a patch scheme $(\beta, \mathcal{A})$, set $\mathbf{b_a} = \mathbf{a}$ for all $\mathbf{a} \in \mathcal{A}$ to get the map

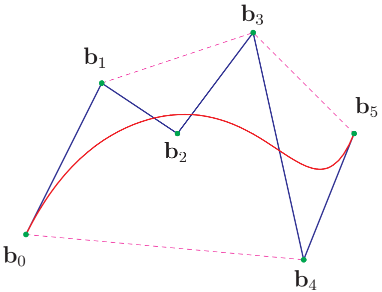
$$\tau : \Delta \longrightarrow \Delta, \qquad x \longmapsto \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{a}.$$

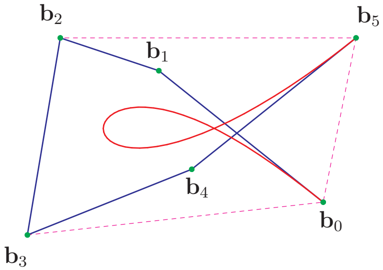
The tautological map τ factors

$$\begin{array}{ccccccc} \tau : \Delta & \xrightarrow{\beta} & & \mathbb{RP}^{\mathcal{A}} & & \xrightarrow{\mu} & \Delta \\ x & \longrightarrow & [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}] & & [y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] & \longrightarrow & \sum_{\mathbf{a} \in \mathcal{A}} y_{\mathbf{a}}(1, \mathbf{a}) \end{array}$$

Write $X_{w,\mathcal{A}}$ for the image $\beta(\Delta)$ in $\mathbb{RP}^{\mathcal{A}}$, which is the positive part of a toric variety. The map $\mu : X_{w,\mathcal{A}} \to \Delta$ is the algebraic moment map.

Injectivity of toric patches





Injectivity of chemical reaction networks

SIAM J. Appl. Math.
Vol. 65, No. 5, pp. 1526–1546

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MULTIPLE EQUILIBRIA IN COMPLEX CHEMICAL REACTION NETWORKS: I. THE INJECTIVITY PROPERTY*

GHEORGHE CRACIUN† and MARTIN FEINBERG‡

- This paper describes a method to decide whether a given reaction network is injective or not.
- Injectivity implies the absence of multiple positive equilibria in these networks.

DETERMINANT EXPANSIONS OF SIGNED MATRICES AND OF CERTAIN JACOBIANS

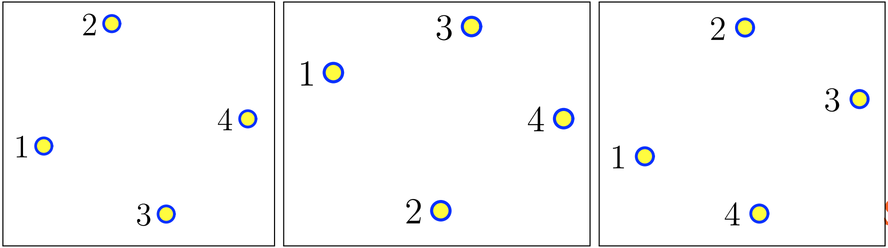
J. WILLIAM HELTON, IGOR KLEP, AND RAUL GOMEZ

Compatible orientations

Let $\mathcal{A}$ and $\mathcal{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\}$ be finite sets of points in $\mathbb{R}^d$.

Suppose that $\{\mathbf{a}_0, \ldots, \mathbf{a}_d\} \subset \mathcal{A}$ and $\{\mathbf{b}_{\mathbf{a}_0}, \ldots, \mathbf{b}_{\mathbf{a}_d}\} \subset \mathcal{B}$ are affinely independent. Then each ordered list determines an orientation.

$\mathcal{A}$ and $\mathcal{B}$ are compatible if either every such pair of orientations is the same, or if every such pair of orientations is opposite.



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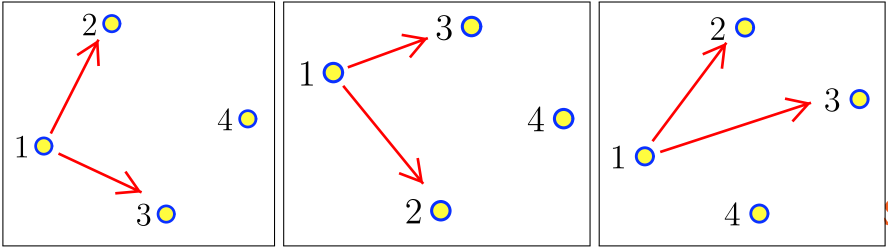

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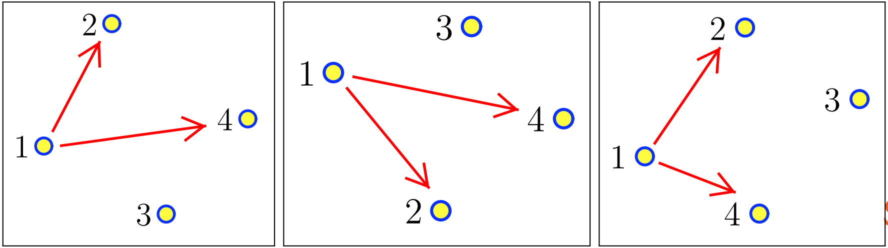

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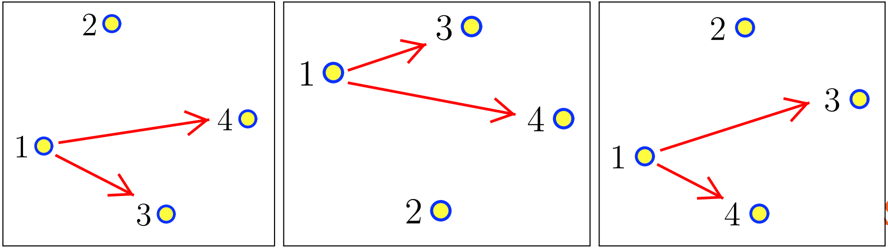


Injectivity of toric patches

For any $w \in \mathbb{R}^{\mathcal{A}}_{>}$, let $F_w \colon \Delta \to \mathbb{R}^d$ be the toric patch of shape $(w, \mathcal{A})$ given by the control points $\mathcal{B} \subset \mathbb{R}^d$:

$$F_w(x) \;:=\; \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(x)\,\mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(x)}\,.$$

Theorem (Craciun-G-Sottile)

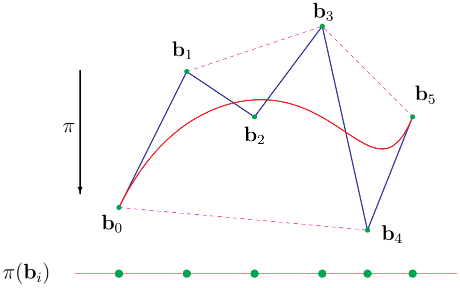
The map F_w is injective for all $w \in \mathbb{R}^{\mathcal{A}}_{>}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible.

The case $\mathbf{b}_{\mathbf{a}} = \mathbf{a}$ for all $\mathbf{a} \in \mathcal{A}$ is known as Birch's Theorem, a fundamental result both for statistics and for toric geometry.

Injectivity of Bézier curves and surfaces

Theorem (Craciun-G-Sottile)

Let $A \subset \mathbb{R}^d$, $w \in \mathbb{R}^{\mathcal{A}}_{>}$, and $\mathcal{B} \subset \mathbb{R}^n$ be the exponents, weights, and control points of a toric patch. If there is a projection $\pi : \mathbb{R}^n - \to \mathbb{R}^d$ such that $\mathcal{A}$ is compatible with $\pi(\mathcal{B})$, then $F_w : \Delta \to \mathbb{R}^n$ is injective.

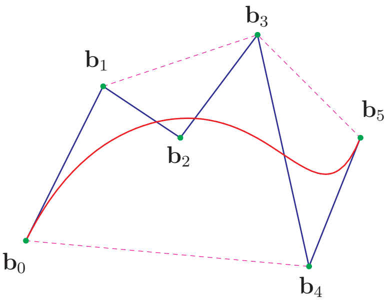


Control polygons

Let $F(t)$ be the Bézier curve given by

$$F(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \mathbf{b_a}, \qquad \text{with } t \in [0,1],$$

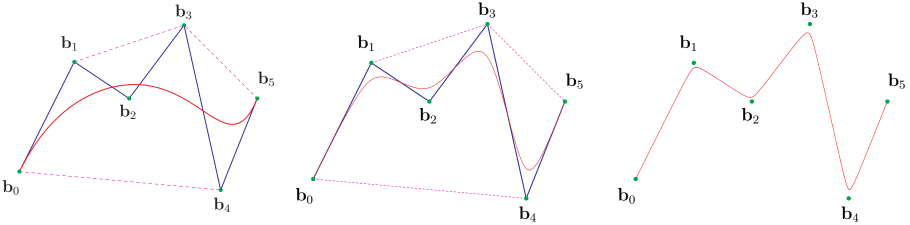
with $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ control points in $\mathbb{R}^\ell$. The corresponding control polygon is the union of the line segments $\overline{\mathbf{b}_0, \mathbf{b}_1}, \overline{\mathbf{b}_1, \mathbf{b}_2}, \ldots, \overline{\mathbf{b}_{n-1}, \mathbf{b}_n}$.



Bézier curve deformations

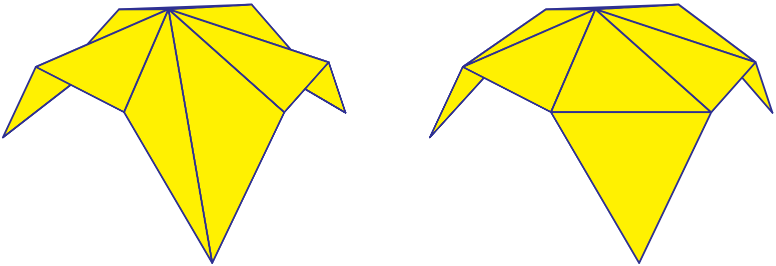
Theorem (Craciun-G-Sottile)

Given control points in $\mathbb{R}^\ell$ and $\epsilon > 0$, there is a choice of weights so that the Bézier curve lies within a distance ϵ of the control polygon.

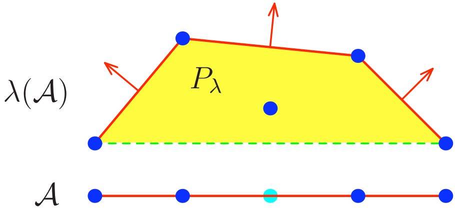


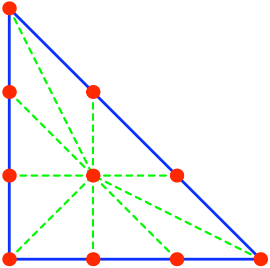
Control polytopes

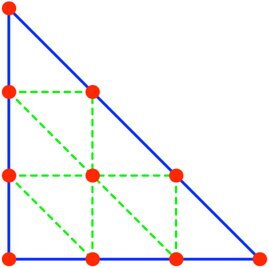
When $F(\Delta)$ is a surface patch, there are many ways to interpolate the control points by triangles to obtain a piecewise linear surface, called a control polytope.



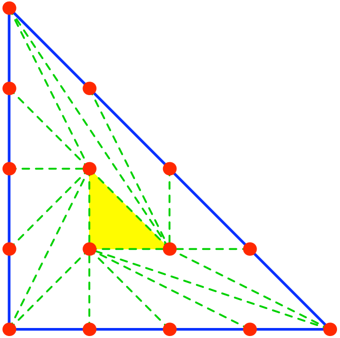
Regular triangulations





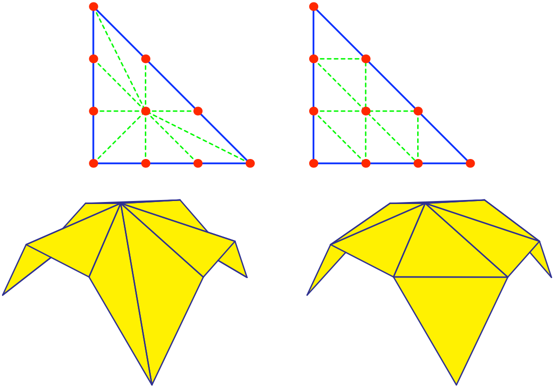


Nonregular triangulation



Regular control polytope

Let $\mathcal{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ be control points indexed by $\mathcal{A} \subset \mathbb{R}^d$ with $d \leq \ell$. Given a regular triangulation $\mathcal{T} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ of $\mathcal{A}$. Define the regular control polytope $\mathcal{B}(\mathcal{T})$ as the union of the simplices $\mathrm{conv}\{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}_i\}$.



Bézier surface deformations

Theorem (Craciun-G-Sottile)

Let $\mathcal{A} \subset \mathbb{R}^d$, $w \in \mathbb{R}^{\mathcal{A}}_{>}$, $\mathcal{B} \subset \mathbb{R}^\ell$ and $\mathcal{T}$ a r. t. of $\mathcal{A}$ induced by λ. For each $t > 0$, let F_t be the toric patch $\left(t^{\lambda(a)} w_{\mathbf{a}}, \mathcal{A}\right)$. Then, for any $\epsilon > 0$ $\exists$ t_0 such that if $t > t_0$, the image $F_t(\Delta)$ lies within ϵ of $\mathcal{B}(\mathcal{T})$.

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