The State Polytope.

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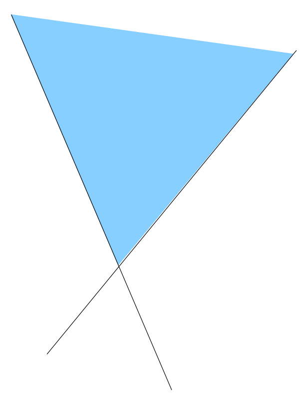
A *polyhedron* is a finite intersection of closed half-spaces in $\mathbb{R}^n$. Thus a polyhedron P can be written as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ where A is a matrix with n columns.

If $b = 0$, then there exist vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ such that

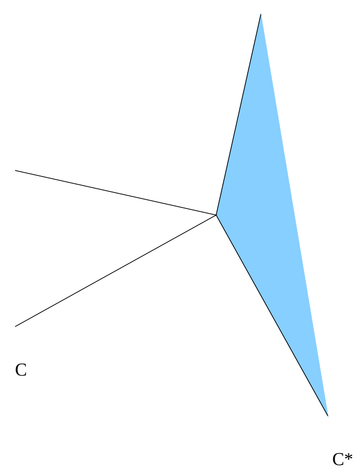
$$\begin{aligned} P &= \mathrm{pos}(u_1, \ldots, u_m) \\ &:= \{\lambda_1 u_1 + \cdots + \lambda_m u_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+\} \end{aligned}$$



A polyhedron of this form is called a *polyhedral cone*.

The *polar* of a cone C is defined as

$$C^* = \{w \in \mathbb{R}^n \mid w \cdot c \leq 0 \; \forall \; c \in C\}.$$



A polyhedron Q which is bounded is called a *polytope*. Every polytope Q can be written as the convex hull of a finite set of points P.

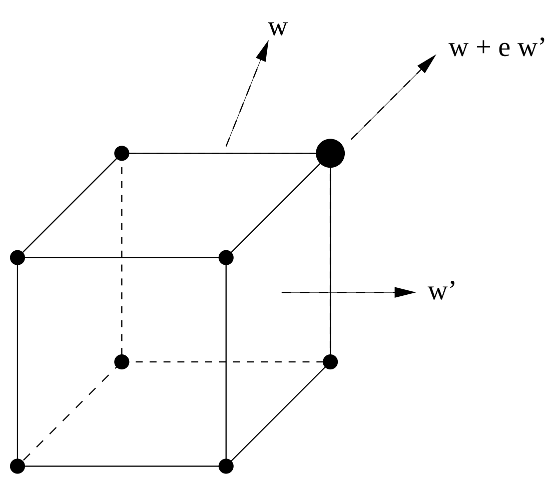
$$\begin{aligned} Q &= \text{conv}(v_1, \ldots, v_m) \\ &:= \{\sum_{i=1}^{m} \lambda_i v_i \mid \; \forall \; \lambda_i \in \mathbb{R}_+, \sum_{i=1}^{m} \lambda_i = 1\}. \end{aligned}$$

Let P be a polyhedron in $\mathbb{R}^n$ and $w \in \mathbb{R}^n$, viewed as a linear functional. We define

$$\text{face}_w(P) := \{u \in P \mid w \cdot u \geq w \cdot v \quad \forall\, v \in P\}.$$

The relation "is a face of" among polyhedra is transitive:

$$\text{face}_{w'}(\text{face}_w(P)) = face_{w+\epsilon w'}(P)$$
$$\text{for } \epsilon > 0 \text{ sufficiently small.}$$

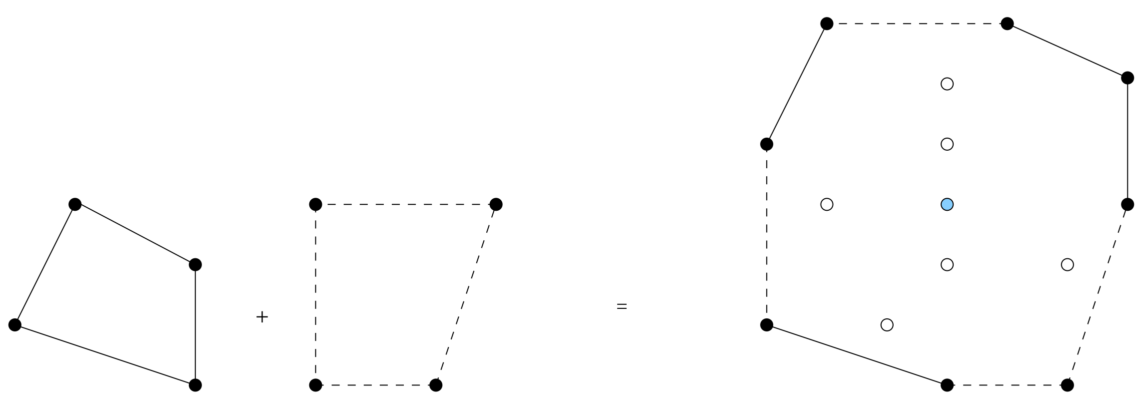


We define the *Minkowski sum* of polyhedra P_1 and P_2 as

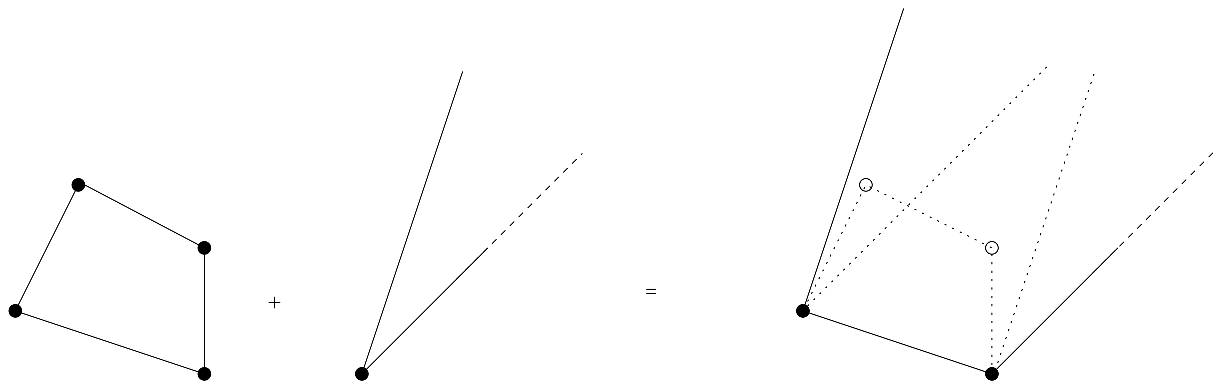
$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}.$$

A basic fact about the Minkowski sum is the additivity of faces:

$$\text{face}_w(P_1 + P_2) = \text{face}_w(P_1) + \text{face}_w(P_2).$$



Proposition 1. *Every polyhedron P can be written as the sum $P = Q + C$ of a polytope Q and a cone C. The cone C is unique and is called the* recession cone *of P.*



A (polyhedral) complex Δ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

(i) If $P \in \Delta$ and F is a face of P, then $F \in \Delta$;

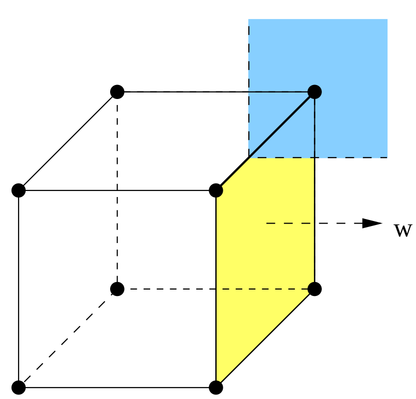
(ii) If $P_1, P_2 \in \Delta$, then $P_1 \cap P_2$ is a face of P_1 and of P_2.

The *support* of a complex Δ is $|\Delta| := \bigcup_{P \in \Delta} P$. A complex Δ which consists of cones is called a *fan*. A fan Δ is *complete* if $|\Delta| = \mathbb{R}^n$.

If $P \subset \mathbb{R}^n$ is a polyhedron and F is a face of P, then the *normal cone* of F at P is

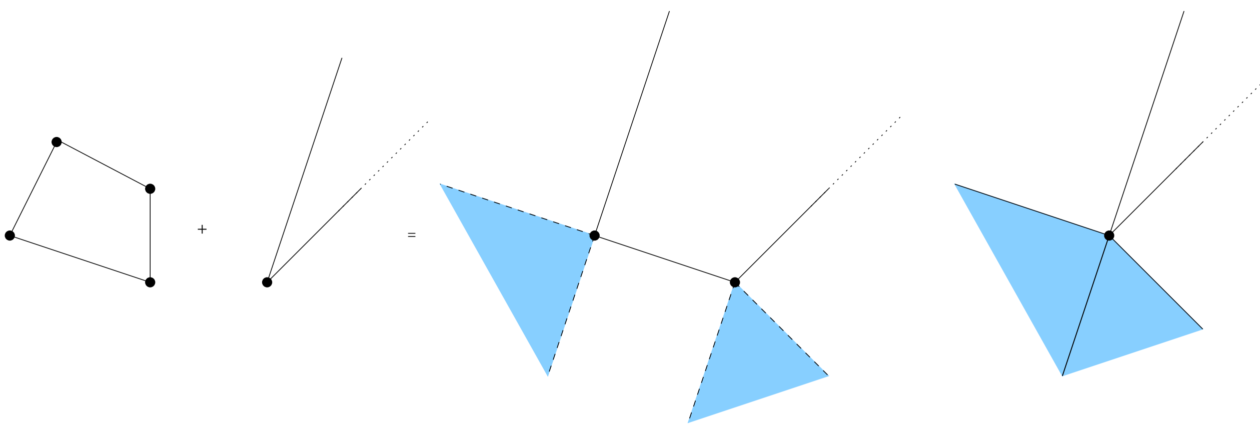
$$\mathcal{N}_P(F) = \{w \in \mathbb{R}^n \mid \text{face}_w(P) = F\}.$$

Note that $\dim(\mathcal{N}_P(F)) = n - \dim(F)$. If F and F' are faces of P, then F' is a face of F if and only if $\mathcal{N}_P(F)$ is a face of $\mathcal{N}_P(F')$.

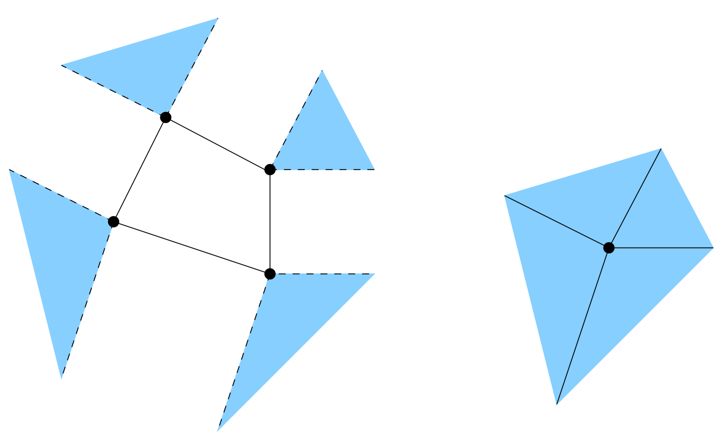


Hence the collection of normal cones $\mathcal{N}_P(F)$, where F ranges over the faces of P, is a fan. This fan is denoted $\mathcal{N}(P)$ and called the *normal fan* of P.

The support of $\mathcal{N}(P)$ equals the polar C^* of the recession cone C.



If Q is a polytope, then its recession cone is $\{0\}$, and its normal fan $\mathcal{N}(Q)$ is a complete fan.



Let $f = \sum_{i=1}^{m} c_i \mathbf{x}^{\mathbf{a}_i}$. The *Newton polytope* of f is defined as $\text{New}(f) := \text{conv}\{\mathbf{a}_i \mid i = 1, \ldots, m\}$ in $\mathbb{R}^n$.

Lemma 2. $\text{New}(f \cdot g) = \text{New}(f) + \text{New}(g)$.

- It suffices to show that both polytopes have the same vertices.

- $\text{face}_w(\text{New}(f)) = \text{New}(\text{in}_w(f))$

- The following relation holds for all w which are sufficiently generic.

$$\begin{aligned}
\text{face}_w(\text{New}(f \cdot g)) &= \text{New}(\text{in}_w(f \cdot g)) \\
&= \text{New}(\text{in}_w(f) \cdot \text{in}_w(g)) \\
&= \text{New}(\text{in}_w(f)) + \text{New}(\text{in}_w(g)) \\
&= \text{face}_w(\text{New}(f)) \\
&\qquad + \text{face}_w(\text{New}(g)) \\
&= \text{face}_w(\text{New}(f) + \text{New}(g)).
\end{aligned}$$

Fix $I \subset k[x]$. Two vectors w, w' are *equivalent* w.r.t. $I \quad :\Longleftrightarrow \ \mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$.

Proposition 3. *Each equivalence class of weight vectors is a relatively open convex polyhedral cone.*

Proof. Let $C[w]$ denote the equivalence class of w. Fix a term order $\prec$. Let G be the reduced Gröbner basis of I w.r.t. $\prec_w$.

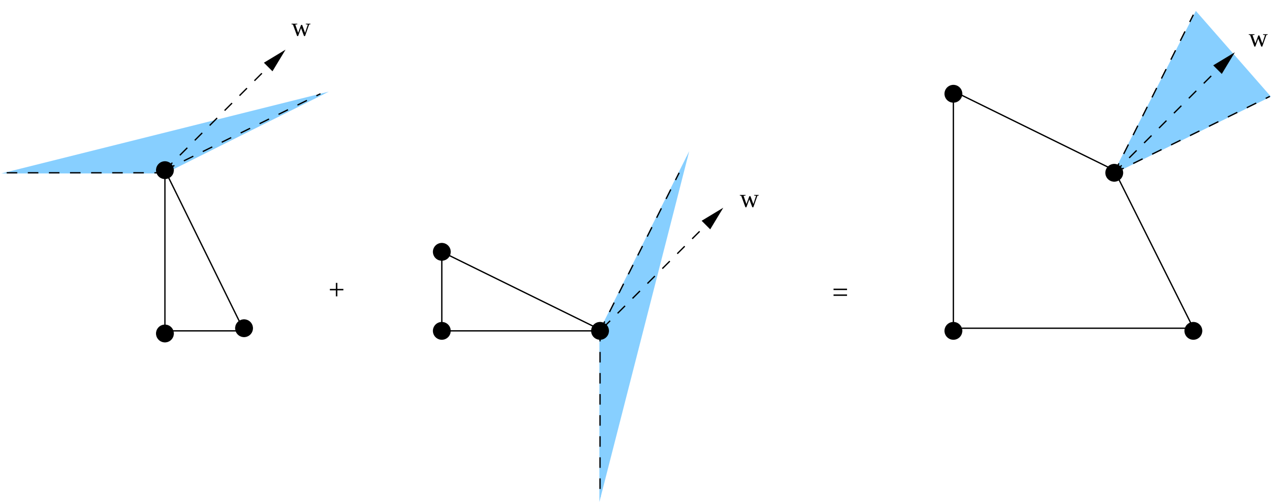
$$C[w] = \{w' \in \mathbb{R}^n \mid \mathrm{in}_{w'}(g) = \mathrm{in}_w(g) \quad \forall\, g \in G\}.$$

This formula expresses $C[w]$ as an intersection of hyperplanes and open half-spaces: $w' \cdot a = w' \cdot b$ and $w' \cdot a > w' \cdot c$, where x^a and x^b run over the terms of $\mathrm{in}_w(g)$ and x^c runs over the terms of g which do not appear in $\mathrm{in}_w(g)$. $\square$

This formula has a geometric reformulation

$$C[w] = \mathcal{N}_Q(\text{face}_w(Q)),$$
$$Q := \text{New}(\prod_{g \in G} g) = \sum_{g \in G} \text{New}(g).$$

Let $I = (x^2 - y + 2, y^2 - x - 3) \subset k[x, y]$. Let $w = (3, 3)$, then $C[w]$ is the following open convex cone:



We define the *Gröbner fan* $\text{GF}(I)$ to be the set of closed cones $\overline{C[w]}$ for all $w \in \mathbb{R}^n$.

From now on we shall assume that I is homogeneous w.r.t. some positive grading $\deg(x_i) = d_i > 0$.

Theorem 4. *There exists a polytope* State(I) *whose normal fan* $\mathcal{N}$(State(I)) *coincides with the Gröbner fan* GF(I).

The polytope State(I) will be called the *state polytope* of I. Its construction goes as follows: Denote by I_d the vector space of homogeneous polynomials of degree d in I. If M is any monomial ideal, then $\sum M_d$ denotes the sum of all vectors $a \in \mathbb{N}^n$ such that x^a has degree d and lies in M.

$$\mathrm{State}_d(I) := \mathrm{conv}\{\textstyle\sum \mathrm{in}_\prec(I)_d \mid \prec \text{ any term order}\}.$$

Let D be the largest degree of any element in a minimal universal Gröbner basis of I.

$$\mathrm{State}(I) := \sum_{d=1}^{D} \mathrm{State}_d(I).$$

We say that a polytope $Q \subset \mathbb{R}^n$ is a state polytope for I if it is strongly isomorphic to State(I). In other words, a polytope Q is a state polytope for I if its normal fan $\mathcal{N}(Q)$ equals the Gröbner fan GF(I).

Proposition 5. *Let* $I = (f)$*, for some homogeneous polynomial* f*. Then* $\mathrm{New}(f)$ *is a state polytope for* I*.*

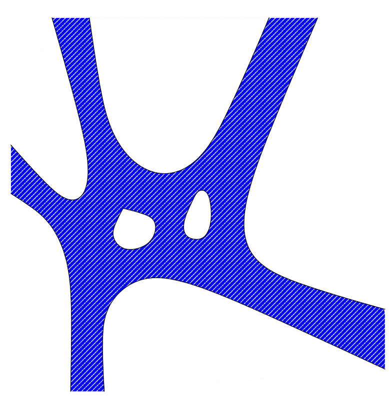
Proof. $\{f\}$ equals the reduced Gröbner basis w.r.t. any term order. Hence
$C[w] = \mathcal{N}_{\mathrm{New}(f)}(\mathrm{face}_w(\mathrm{New}(f)))$. Thus the equivalence classes of term orders are the normal cones of the Newton polytope $\mathrm{New}(f)$. □

Corollary 6. *Let* G *be a universal Gröbner basis of* I *which is a reduced Gröbner basis of* I *w.r.t. every term order. Then* $\sum_{g \in G} \mathrm{New}(g)$ *is a state polytope for* I*.*

Some spectacular applications of Newton polytopes to classical algebraic problems have been found by A. Kouchnirenko, Bernstein, Khovansky, Gelfand, Krapanov, Zelevinsky, Sturmfels, and many others. The structure of New(f) is deeply related to the geometry of the hypersurface $\{f = 0\}$. Denote by log : $(\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ the map

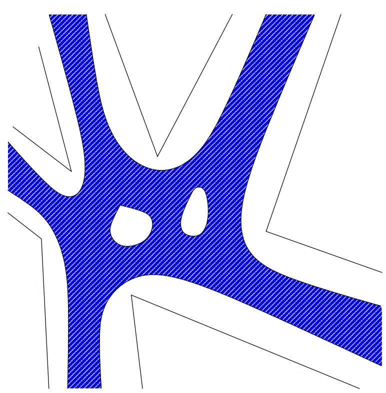
$$(x_1, \ldots, x_n) \longmapsto (\log |x_1|, \ldots, \log |x_n|).$$

For a polynomial $f \in k[x]$, denote by Z_f the hypersurface in $(\mathbb{C}^*)^n$, defined by the equation $f = 0$. The *amoeba* of f is the subset $\log(Z_f) \subset \mathbb{R}^n$.



Theorem 7. *The vertices of* New(f) *are in bijection with those connected components of the complement* $\mathbb{R}^n \setminus \log(Z_f)$ *which contain a convex cone with non-empty interior.*

The normal cone $\mathcal{N}(Q)$ is complete, so the amoeba is situated in thin spaces between walls of the translated normal cones. It follows that the combinatorial structure of the Newton polytope New(f) can be read from the geometry of the hypersurface Z_f.



Suppose we have k polynomials $f_1, \ldots, f_k$ in k variables. These polynomials define functions on the algebraic torus $(\mathbb{C}^*)^k$. We want to find the number of their common roots in this torus.

Theorem 8. *Let $A_1, \ldots, A_k \subset \mathbb{Z}^k$ be finite sets such that $\bigcup_{i=1}^{k} A_i$ generates $\mathbb{Z}^k$ as an affine lattice. Let $Q_i = \text{conv}(A_i)$, and let $\mathbb{C}^{A_i}$ be the space of polynomials in $x_1, \ldots, x_k$ with monomials from A_i. Then there exists a dense Zariski open subset $U \subset \prod \mathbb{C}^{A_i}$ with the following property: for any $(f_1, \ldots, f_k) \in U$, the number of solutions of the system of equations $f_1(x) = \cdots = f_k(x) = 0$ in $(\mathbb{C}^*)^k$ equals the mixed volume $\text{vol}_{\mathbb{Z}^k}(Q_1, \ldots, Q_k)$.*

Observe that each Q_i is the Newton polytope of a generic $f \in \mathbb{C}^{A_i}$.