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Sequential dynamical systems over words

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Abstract

This paper is motivated by the theory of sequential dynamical systems (SDS), developed as a basis for a mathematical theory of computer simulation. A sequential dynamical system is a collection of symmetric Boolean local update functions, with the update order determined by a permutation of the Boolean variables. In this paper, the notion of SDS is generalized to allow arbitrary functions over a general finite field, with the update schedule given by an arbitrary word on the variables. The paper contains generalizations of some of the known results about SDS with permutation update schedules. In particular, an upper bound on the number of different SDS over words of a given length is proved and open problems are discussed.

Keywords: Sequential dynamical systems; Galois correspondence; Words; Dependency graph; Acyclic orientation; Dynamically equivalent systems

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1. Introduction

The theory of sequential dynamical systems (SDS) was introduced in [1–4] as a way to formalize certain types of large-scale computer simulations. Such a mathematical formalization is crucial in effective software verification and can aid in simulation output analysis. The study of SDS, while motivated by these applications, leads to a wide variety of mathematical problems of intrinsic interest, which combines several mathematical areas, ranging from dynamical systems theory to combinatorics and algebra. This paper generalizes some of the original SDS concepts and results.

Sequential dynamical systems, as defined in [1–4], incorporate the essential features of interaction-based (also sometimes called agent-based) computer simulations. Local variables $v_1,\ldots,v_n$ take on binary states which evolve in discrete time, based on a local update function f^i attached to each variable v_i, and which depends on the states of certain other variables, encoded by the edges of a *dependency graph* Y on the vertices $v_1,\ldots,v_n$. Finally, an update schedule prescribes how these local update functions are to be composed in order to generate a global update function

$$f:\ \{0,1\}^n \to \{0,1\}^n$$

of the system. An important question, which can be answered in this setting, is how many different systems one can generate simply by varying the update schedule. The upper bound is given in terms of invariants of the dependency graph Y.

In applications the dependency graph Y frequently varies over time, however. The need for a framework that allows for such a change inspired the investigation of properties of tuples of "local" functions in [5] and certain equivalence relations on them. That paper also contains a Galois correspondence between sets of tuples of local functions and certain graphs. Tuples of local functions can be interpreted as parallel update systems $f:\ \{0,1\}^n \to \{0,1\}^n$, so that the results pertain to the study of parallel update systems as well. The present paper makes the connection between tuples of local functions (parallel systems) and sequential systems, in particular SDS, explicit by exploiting this Galois correspondence. In order to describe our main results, we need to recall some definitions and results from Laubenbacher and Pareigis [5] in the following section.

Then we focus on tuples $(f^1,\ldots,f^n)$ of functions $f^i:\mathbb{K}^n \to \mathbb{K}$, with $\mathbb{K}$ a finite field and consider systems $f:\ \mathbb{K}^n \to \mathbb{K}^n$ which are obtained by composing local functions from such an n-tuple $f = (f^1,\ldots,f^n)$. We show that, if $t \geqslant 1$ is an integer and W_t is the set of all words in the integers $1,\ldots,n$ of length t, allowing for repetitions, then we obtain an upper bound for the number of different systems $f^\pi:\mathbb{K}^n \to \mathbb{K}^n$ one can construct by forming

$$f^{\pi} = f^{i_t} \circ \cdots \circ f^{i_1},$$

where $\pi = (i_1, \ldots, i_t)$ ranges over all elements in W_t.

This upper bound generalizes one for SDS, derived in [6]. It suggests that a part of the theory of SDS can be derived for systems that are SDS-"like," but have fewer restrictions on the local functions. In particular, it is not necessary to make the dependency graph an explicit part of the data defining an SDS. This approach is explored further in [7], in which a more general notion of SDS is introduced, and transformations of SDS are defined, forming a category with interesting properties, that contains SDS as a subcategory. A transformation between two SDS can be viewed as a simulation of one system by the other.

2. Parallel systems

Throughout this paper $\mathbb{K}$ denotes a finite field of order q and $\mathbb{K}^n$ is the n-fold cartesian product of $\mathbb{K}$.

Definition 2.1. Let n be a positive integer, let d be a non-negative integer, and let Y be a graph with vertex set $\{1, \ldots, n\}$.

(1) A function $f: \mathbb{K}^n \to \mathbb{K}^n$ is *d-local* on Y if, for any $1 \leqslant j \leqslant n$, the jth coordinate of the value of f on $x \in \mathbb{K}^n$ depends only on the values of those coordinates of x that have distance less than or equal to d from vertex $j \in Y$. In other words, if $f(x) = (f_1(x), \ldots, f_n(x))$, then $f_j: \mathbb{K}^n \to \mathbb{K}$ depends only on those coordinates that have distance less than or equal to d from j.
(2) For $1 \leqslant d < n$ and $1 \leqslant j \leqslant n$, let $L_d^j(Y)$ be the set of all functions $f: \mathbb{K}^n \to \mathbb{K}^n$ such that
$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, f_j(x), x_{j+1}, \ldots, x_n)$$
and $f_j: \mathbb{K}^n \to \mathbb{K}$ depends only on the values of those coordinates of x which have distance at most d from j in Y. Hence $L_d^j(Y)$ consists of d-local functions on $\mathbb{K}^n$, which are the identity on all but possibly the jth coordinate.
(3) For $d = n$, define $L_n^j(Y)$ to be the set of all functions on $\mathbb{K}^n$, which are the identity on all but possibly the jth coordinate. Observe that if Y is connected, then this definition of $L_n^j(Y)$ directly extends the definition in (2).

Observe that $L_0^j(Y) = L_0^j$ does not depend on the graph Y and, if Y is connected, neither does $L_n^j(Y)$. Furthermore, L_0^j is isomorphic to $\mathrm{Map}(\mathbb{K}, \mathbb{K}) = \{f: \mathbb{K} \to \mathbb{K}\}$. For $\mathbb{K} = \{0, 1\}$, the set L_0^j contains all four possible functions, namely the identity on $\mathbb{K}$, the two projections to one element in $\mathbb{K}$, and the inversion.

In this paper we study the set

$$L_n^1 \times \cdots \times L_n^n = \{(f^1, \ldots, f^n) | f^i \in L_n^i\},$$

that is, the set of n-tuples of functions $f^i : \mathbb{K}^n \to \mathbb{K}^n$ which only change the ith coordinate. To be precise, $f^i(x) = (x_1, \ldots, x_{i-1}, f_i^i(x), x_{i+1}, \ldots, x_n)$, with arbitrary functions $f_i^i : \mathbb{K}^n \to \mathbb{K}$. We denote by $\mathscr{F}$ the power set of this set without the empty set. The following theorem is one of the main results in [5].

Theorem 2.2. *There is a Galois correspondence between $\mathscr{F}$ and the set $\mathscr{G}$ of subgraphs of the complete graph K_n on the vertex set $\{1, \ldots, n\}$.*

For the convenience of the reader we recall the construction of this Galois correspondence. Let $F \in \mathscr{F}$. Define a subgraph $\Phi(F)$ of the complete graph K_n as follows. First construct the set $\widetilde{F}$ of all n-tuples $\widetilde{f} = (\widetilde{f^1}, \ldots, \widetilde{f^n})$, which either are in F or arise from an element in F by replacing one of the coordinates by a 0-local function, that is, by a function from L_0^i for some i. Now define the graph $\Phi(F)$ as follows. An edge (i,j) of K_n is in $\Phi(F)$ if and only if $\widetilde{f^i} \circ \widetilde{f^j} = \widetilde{f^j} \circ \widetilde{f^i}$ for all $\widetilde{f} = (\widetilde{f^1}, \ldots, \widetilde{f^n}) \in \widetilde{F}$.

Conversely, let $G \subset K_n$ be a subgraph. We define a set $\Psi(G)$ of n-tuples of functions on $\mathbb{K}^n$ by

$$\Psi(G) = L_1^1(\overline{G}) \times L_1^2(\overline{G}) \times \cdots \times L_1^n(\overline{G}),$$

where $\overline{G}$ is the complement of G in K_n. Then Φ and Ψ together form the desired Galois correspondence.

In particular, if $F \in \mathscr{F}$ consists of one element $f = (f^1, \ldots, f^n)$, then the graph $\Phi(\{f\}) = \Phi(f)$ encodes the dependency relations among the local functions f^i. Conversely, for a subgraph $G \subset K_n$, the set $\Psi(G)$ contains all n-tuples of local functions whose dependency relations are modeled by G. The graph $\Phi(f)$ is an essential part of the definition of an SDS, as in [1–4]. In the following section, we give an algebraic criterion for the computation of $\Phi(f)$, based on the fact that any local function f_i^i can be presented uniquely as a polynomial.

3. Computation of the dependency graph Φ

In this section, we give a method for computing the graph $\Phi(f)$ for an n-tuple of local functions f. It relies on the fact that any local function $g : \mathbb{K}^n \to \mathbb{K}$ can be represented uniquely as a polynomial h on n variables [8]. Suppose $|\mathbb{K}| = q$ and let $h : \mathbb{K}^n \to \mathbb{K}$ be the polynomial defined by

$$h(x_1, \ldots, x_n) = \sum_{(c_1, \ldots, c_n) \in \mathbb{K}^n} \left[g(c_1, \ldots, c_n) \prod_{i=1}^{n} (1 - (x_i - c_i)^{q-1}) \right].$$

It is easy to see that $h(c_1,\dots,c_n) = g(c_1,\dots,c_n)$ for all $(c_1,\dots,c_n) \in \mathbb{K}^n$. Therefore, any function in L_n^j can be written uniquely as an n-tuple of polynomials in n variables $x_1,\dots,x_n$.

Let $f = (f^1,\dots,f^n):\ \mathbb{K}^n \to \mathbb{K}^n$ be an element of $L_n^1 \times \cdots \times L_n^n$ and let $\Phi(f)$ be its associated graph. The following result gives an important characterization of $\Phi(f)$ in terms of the polynomial representation of the entries of f.

Proposition 3.1. *There is an edge between vertex i and vertex j in $\Phi(f)$ if and only if x_i does not appear in f_j^j and x_j does not appear in f_i^i.*

Proof. Suppose first that x_i does not appear in f_j^j and x_j does not appear in f_i^i. It is clear that then f^i and f^j commute. Now consider an $\widetilde{f}$ which is obtained from f by replacing the jth coordinate by a 0-local function $\widetilde{f}^j$. It also does not depend on x_i. Similarly, no $\widetilde{f^i}$ depends on x_j. Thus, $\widetilde{f^j} \circ \widetilde{f^i} = \widetilde{f^i} \circ \widetilde{f^j}$ for all $\widetilde{f} = (\widetilde{f^1},\dots,\widetilde{f^n})$ obtained from f by replacing one of the coordinates by a 0-local function. Hence the edge (i,j) is in the graph $\Phi(f)$.

Conversely, suppose that x_i divides a monomial of f_j^j or x_j divides a monomial of f_i^i. Without loss of generality, suppose x_i divides a monomial of f_j^j. Let $\mathscr{M} = \{m_1,\dots,m_t\}$ be the set of all monomials of f_j^j such that $x_i | m_l$ for all $l = 1,\dots,t$. Let m_s be a monomial in $\mathscr{M}$ of minimal degree, say $m_s = x_{s_1}^{t_1} x_{s_2}^{t_2} \cdots x_{s_r}^{t_r}$, where $x_i \in \{x_{s_1},\dots,x_{s_r}\} \subseteq \{x_1,\dots,x_n\}$. Define $a = (a_1,\dots,a_n) \in \mathbb{K}^n$ by setting, for $1 \leqslant l \leqslant n$,

$$a_l = \begin{cases} 1 & \text{if } l \in \{s_1,\dots,s_r\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the jth coordinate of $f^j(a)$ is equal to

$$f_j^j(a) = (m_1 + \cdots + m_t)(a) = m_s(a) = 1.$$

Let $\widetilde{f^i}$ be the 0-local function, projection to zero, in the ith coordinate. The jth coordinate of $\widetilde{f^i} \circ f^j(a)$ is equal to

$$\left(\widetilde{f^i} \circ f^j(a)\right)_j = f_j^j(a) = 1.$$

On the other hand, $\widetilde{f^i}(a) = (a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n)$. So the jth coordinate of $f^j \circ \widetilde{f^i}(a)$ is

$$\begin{aligned}\left(f^j \circ \widetilde{f^i}(a)\right)_j &= f_j^j(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n)\\ &= (m_1 + \cdots + m_t)(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n) = 0.\end{aligned}$$

Thus $(\widetilde{f^i} \circ f^j(a))_j \neq (f^j \circ \widetilde{f^i}(a))_j$. Therefore, $\widetilde{f^i} \circ f^j \neq f^j \circ \widetilde{f^i}$, and hence there cannot be an edge between vertices i,j in $\Phi(f)$. $\square$

This proposition provides an easy algorithm to compute $\Phi(f)$ for any f, or, more generally, any set of such functions. (A C++ implementation is available from the authors.)

4. Sequential systems over words

In many cases, it is necessary to study systems that are updated asynchronously rather than in parallel. So it is important to study systems $f: \mathbb{K}^n \to \mathbb{K}^n$ obtained by composing the local functions of an n-tuple according to some prescribed update schedule, rather than the tuple itself. A theoretical question which has important practical consequences is how many different systems one can obtain by simply varying the update schedule of the variables, that is, by composing the local functions in a different order. In this section, we derive an upper bound for this number.

Definition 4.1. Let $f = (f^1, \ldots, f^n): \mathbb{K}^n \to \mathbb{K}^n$, and let W_t be the set of all words on $\{1,\ldots,n\}$ of length t, for some $t \geqslant 1$, allowing for repetitions. For $\pi = (i_1,\ldots,i_t) \in W_t$, we denote by f^π the finite dynamical system given by

$$f^{i_t} \circ \cdots \circ f^{i_1} : \mathbb{K}^n \to \mathbb{K}^n.$$

Let $F_{W_t}(f) = \{f^\pi | \pi \in W_t\}$, the collection of all systems $\mathbb{K}^n \to \mathbb{K}^n$ that can be obtained by composing the coordinate functions of f in all possible ways, using up to t of them.

We now define an equivalence relation on W_t.

Definition 4.2. Let G be a graph on the n vertices $1,\ldots,n$. Let $\sim_G$ be the equivalence relation on W_t generated by the following relation. Let $\pi = (i_1,\ldots,i_t) \in W_t$. For $1 \leqslant s < t$, if $i_s = i_{s+1}$ or there is no edge between i_s and i_{s+1} in G, then

$$\pi \sim_G \pi',$$

where $\pi' = (i_1,\ldots,i_{s+1}, i_s, i_{s+2},\ldots,i_t)$.

Remark 4.3. If $f = (f^1, \ldots, f^n): \mathbb{K}^n \to \mathbb{K}^n$ is such that $\Phi(f) = G$, and $\pi \sim_{\overline{G}} \pi'$, then

$$f^{i_t} \circ \cdots \circ f^{i_s} \circ f^{i_{s+1}} \circ \cdots \circ f^1 = f^{i_t} \circ \cdots \circ f^{i_{s+1}} \circ f^{i_s} \circ \cdots \circ f^1.$$

We now derive an upper bound on the size of the set $F_{W_t}(f)$, that is, on the number of different systems one obtains by composing the coordinate functions of f in all possible orders, with up to t of them at a time.

Definition 4.4. Let $f = (f^1, \ldots, f^n) : \mathbb{K}^n \to \mathbb{K}^n$, and let $G(f) = G = \Phi(f)$. Let $\pi = (i_1, \ldots, i_t) \in W_t$. Let $H_\pi(f) = H_\pi$ be the graph on t vertices $v_1, \ldots, v_t$, corresponding to $i_1, \ldots, i_t$ (with $v_a \neq v_b$ even in the case that $i_a = i_b$), with an edge between v_a and v_b if and only if the following two conditions hold:

(1) $i_a \neq i_b$,
(2) the edge (i_a, i_b) is not in G.

Remark 4.5. Observe that if $\pi \in S_n$, that is, $t = n$ and π contains no repetitions, then $H_\pi = \overline{G}$.

Let Acyc(H) be the set of all acyclic orientations of a graph H. Given $\pi = (i_1, \ldots, i_t) \in W_t$, we construct an acyclic orientation of H_π by orienting an edge (v_i, v_j) toward the vertex whose label occurs first in π. If all entries of π are distinct, then this clearly produces an acyclic orientation. But even if an entry is repeated we cannot produce an oriented cycle, since there is no edge between the vertices corresponding to the repetitions. Denote this acyclic orientation by $\mathcal{O}_\pi(f)$.

Lemma 4.6. *If* $\pi \sim_{\overline{G}} \pi'$, *then* $H_\pi(f) = H_{\pi'}(f)$ *and* $\mathcal{O}_\pi(f) = \mathcal{O}_{\pi'}(f)$.

Proof. If $\pi \sim_{\overline{G}} \pi'$, then they differ by a sequence of transpositions of adjacent letters, which are either equal, or for which the corresponding vertices in G are connected by an edge. Hence $H_\pi(f)$ and $H_{\pi'}(f)$ have the same vertex set. Furthermore, an edge (a, b) is in $H_\pi(f)$ if and only if $i_a \neq i_b$ and (i_a, i_b) is an edge in G; similarly for $H_{\pi'}(f)$. Observe that the transposition in π of adjacent letters which are connected by an edge in G does not change the resulting acyclic orientation, because, by construction, the vertices v_a and v_b are not connected by an edge in $H_\pi(f)$. Hence the proof of the lemma is complete. $\square$

The next proposition is a generalization of a result from Cartier–Foata normal form theory. See, e.g., [9,6].

Proposition 4.7. *Let f be a system and $G = \Phi(f)$. There is a one-to-one correspondence*

$$\psi_G : W_t/\sim_{\overline{G}} \to \{Acyc(H_\pi(f)) | \pi \in W_t\}.$$

Proof. We assign to a word $\pi \in W_t$ the associated acyclic orientation $\mathcal{O}_\pi(f)$ on $H_\pi(f)$. By Lemma 4.6 this induces a mapping ψ_G on $W_t/\sim_{\overline{G}}$. There is an obvious inverse mapping, assigning to an acyclic orientation on H_π the corresponding π', equivalent to π, such that i_a appears before i_b in π' if there is an edge (a, b) in $H_\pi(f)$, oriented from a to b. $\square$

Example 4.8. We illustrate this correspondence with the following example. Let G be a 4-cycle with vertices 1,...,4, and let $\pi = (1,2,1,3)$. Then H_π has the four vertices 1,11,2,3, where 11 represents the vertex corresponding to the second 1 in π. There is an edge $3 \to 1$, which becomes oriented toward 1 in the acyclic orientation $\mathcal{O}_\pi$.

The following theorem provides an upper bound on the number of different systems $f^\pi : \mathbb{K}^n \to \mathbb{K}^n$ one can obtain from composing the coordinate functions of an n-tuple $f = (f^1,\ldots,f^n)$, up to t of them at a time.

Theorem 4.9. *Let $f = (f^1,\ldots,f^n)$ be a system of local functions on $\mathbb{K}^n$, and let $F_{W_t}(f) = \{f^\pi | \pi \in W_t\}$. Then*

$$|F_{W_t}(f)| \leqslant |\{Acyc(H_\pi(f)) | \pi \in W_t\}| = \sum_{\pi \in W_t} |\{Acyc(H_\pi(f))\}|.$$

Proof. By Proposition 4.7, $|W_t/\sim_{\overline{G}}| = |\{\mathrm{Acyc}(H_\pi(f)) : \pi \in W_t\}|$. But we have seen that if $\pi \sim_{\overline{G}} \pi'$ then $f^\pi = f^{\pi'}$. Hence

$$|F_{W_t}(f)| \leqslant |W_t/\sim_{\overline{G}}| = |\{\mathrm{Acyc}(H_\pi(f)) : \pi \in W_t\}|. \qquad \square$$

The following example shows that the upper bound in the theorem is not attained in general.

Example 4.10. Let $f = (x_2x_3, x_1x_3, 0) : \mathbb{K}^3 \to \mathbb{K}^3$. Then $\Phi(f)$ does not contain any edges. Let $\pi = (3,2,1)$, $\pi' = (3,1,2)$. Then $\pi \nsim_{\overline{G}} \pi'$. However,

$$f^\pi = f^1 \circ f^2 \circ f^3 = 0 = f^2 \circ f^1 \circ f^3 = f^{\pi'}.$$

Corollary 4.11. *If $\pi \in S_n$, then $H_\pi(f) = \overline{G}$ and hence $\{Acyc(H_\pi(f)) : \pi \in S_n\} = \{Acyc(\overline{G})\}$. Thus, in this case, we recover the upper bound for the number of different SDS obtained in [2].*

If we restrict ourselves to SDS, this bound is known to be sharp [6]. For general systems, however, it is not sharp, since the number of all possible systems on $\mathbb{K}^n$ is bounded above by q^{nq^n}, but one can always find a word π on n letters long enough, such that $|\mathrm{Acyc}(H_\pi)| > q^{nq^n}$.

Open Problem 1. Let $f = (f^1,\ldots,f^n)$ be a system of local functions on $\mathbb{K}^n$, let W_t be the set of all words of length t on n letters, and let $F_{W_t}(f) = \{f^\pi | \pi \in W_t\}$. Find a sharp upper bound for $|F_{W_t}(f)|$.

In [4,10], an upper bound for dynamically non-equivalent SDS is given. In general, two maps $f, g : \mathbb{K}^n \to \mathbb{K}^n$ are *dynamically equivalent* if there exists a bijection $\varphi : \mathbb{K}^n \to \mathbb{K}^n$ such that

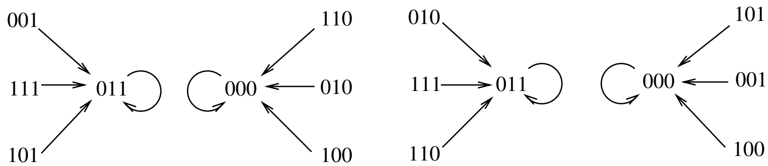

Fig. 1. The state spaces of $f^3 \circ f^2 \circ f^1$ and $f^1 \circ f^2 \circ f^3$.

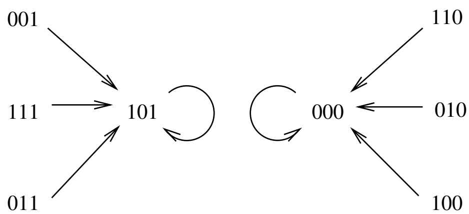

Fig. 2. The state space of $\varphi \circ f^3 \circ f^2 \circ f^1 \circ \varphi^{-1}$.

$$g = \varphi \circ f \circ \varphi^{-1}.$$

In this case, we write, $f \equiv g$. The upper bound in [4,10] relies on the fact that conjugacy yields an SDS with the same graph and local functions. This is not true for the general systems discussed in this paper as the following example shows. Thus, this upper bound holds exactly for the class of SDS.

Example 4.12. Let $f = (0, x_3, x_2) : \mathbb{K}^3 \to \mathbb{K}^3$. Then $\Phi(f)$ is the graph on three vertices 1,2,3 with edges (1,2),(1,3). Hence there are only two functionally non-equivalent systems which correspond to the permutations id = (123) and (321), that is, the systems $f^3 \circ f^2 \circ f^1$ and $f^1 \circ f^2 \circ f^3$. These two systems have the state spaces in Fig. 1. Nevertheless, if we let $\varphi = (213)$, then the system $\varphi \circ f^{\mathrm{id}} \circ \varphi^{-1}$ has the state space in Fig. 2. As expected, this state space is isomorphic to the state space of f^{id} but it is not equal to any of the two possible state spaces given in Fig. 1. In fact, it is not equal to the state space of any system obtained by composing the functions f^1, f^2, f^3 according to any word. This is easily seen because, for example, the state (1,0,1) is sent to itself, but f^1 is the zero function. So f^1 cannot be involved. On the other hand, the first coordinate of other states changes, so there must be a function involved that changes the first coordinate.

Open Problem 2. Let $f = (f^1, \ldots, f^n)$ be a system of local functions on $\mathbb{K}^n$, W_t be the set of all words of length t on n letters, and let $F_{W_t}(f) = \{f^\pi | \pi \in W_t\}$. Find a sharp upper bound for $|F_{W_t}(f)/\equiv|$.

5. Discussion

The approach to the study of finite systems taken in this paper was originated in [5], motivated by the desire to better understand sequential dynamical systems.

Recall that an SDS $f: \mathbb{K}^n \to \mathbb{K}^n$ is given by a graph Y with n vertices, functions $f^i: \mathbb{K}^n \to \mathbb{K}^n$, which change only the ith coordinate and take as input those coordinates connected to v_i in the graph Y. These functions are then composed according to an update schedule given by a permutation $\pi \in S_n$. That is,

$$f = f^{\pi(n)} \circ \cdots \circ f^{\pi(1)}.$$

The functions f^i are required to be symmetric in their inputs, that is, permuting the inputs does not change the value of the function.

In this paper, we study n-tuples of functions $f^i: \mathbb{K}^n \to \mathbb{K}^n$, which change only the ith coordinate, without any further restrictions. In particular, we do not suppose the a priori existence of a graph Y, that governs the dependencies among these functions. The Galois correspondence Ψ, constructed in [5], provides such a graph when needed. And, as for SDS, it is the invariants of this graph that determine many properties of the n-tuple and finite systems derived from it. It shows that even for SDS it is not necessary to explicitly include the dependency graph Y in the data defining an SDS. The Galois correspondence also shows that in general there will be more than one system whose dependency relations are modeled by a given graph.

An important theoretical result, proved in [6], gives a sharp upper bound on the number of different SDS that can be obtained by varying the update schedule over all of S_n. The proof of this result assumes that all vertices of Y that have the same degree also have the same local function attached to them. Theorem 4.9 generalizes this upper bound by removing the restrictions on the local functions f^i and on the graph Y. More importantly, it removes the restriction that the update schedule be given by a permutation. Thus, the upper bound holds for compositions of the coordinate functions, which allows for repetitions of the functions, and does not require that all functions are actually used.

These generalizations suggest that a more relaxed definition of SDS can still lead to a class of systems about which one can prove theorems like the above upper bound. Such a definition is proposed in [7], where a category of more general SDS is developed.

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