Minimal Cohen–Macaulay Deformations of Matroid Ideals

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Cellular Resolutions

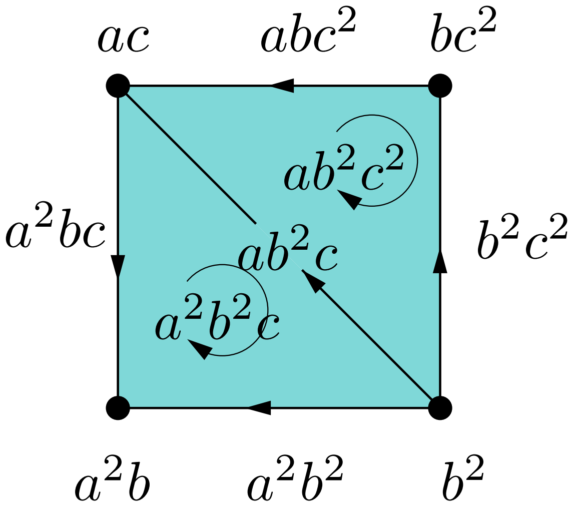
- Let $M = \langle m_1, m_2, \ldots, m_r \rangle$ be a monomial ideal in $S = S = \mathbf{k}[x_1, x_2, \ldots, x_n]$.
- Let X be a finite regular CW-complex with r vertices.
- Label each vertex of X by the generators of M, and each face of X by the lcm of the labels of its vertices.
- Fix an orientation on X.
- The Cellular complex $\mathbb{F}_X$ supported on X is the complex of $\mathbb{Z}^n$–graded modules

$$\mathbb{F}_X = \bigoplus_{F \in X} S[-\mathbf{a}_F],$$

$$\partial(F) = \sum_{\text{facets } G \text{ of } F} \operatorname{sign}(G, F) \frac{\mathbf{x}^{\mathbf{a}_F}}{\mathbf{x}^{\mathbf{a}_G}} G$$

Cellular Resolutions — Example

- Let $M = \langle a^2b, ac, b^2, bc^2 \rangle$ be an ideal in $S = \mathbf{k}[a, b, c]$.
- Let X be the finite regular CW-complex:



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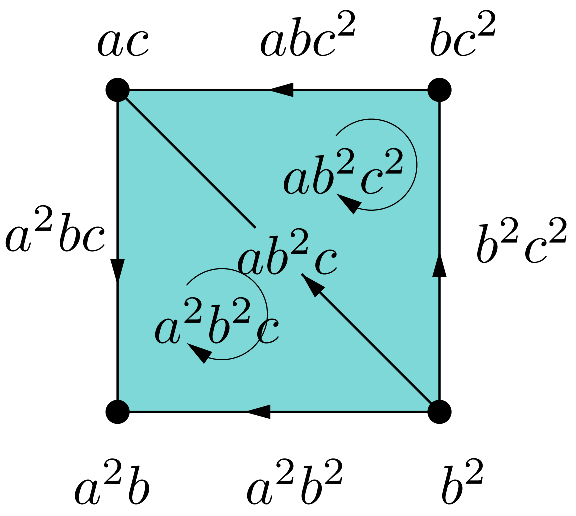
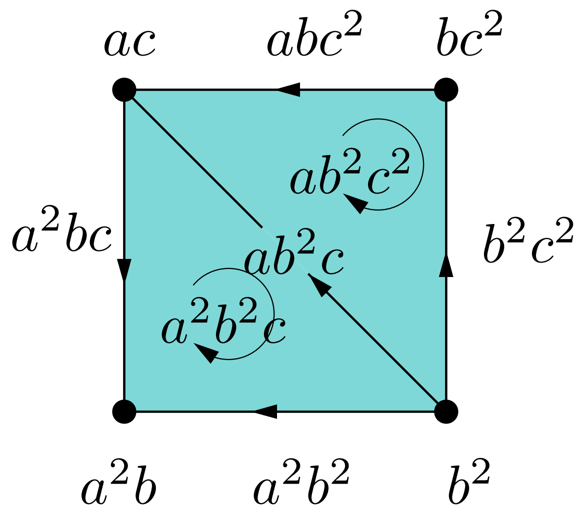


- Let $\mathbb{F}_X$ be the complex of $\mathbb{Z}^n$-graded free S-modules

$$0 \to S[-(2,2,1)] \oplus S[-(1,2,2)] \xrightarrow{\partial_2} S[-(2,1,1)] \oplus S[-(2,2,0)] \oplus$$

$$S[-(1,2,1)] \oplus S[-(1,1,2)] \oplus S[-(0,2,2)] \xrightarrow{\partial_1} S[-(2,1,0)] \oplus$$

$$S[-(1,0,1)] \oplus S[-(0,1,2)] \oplus S[-(0,2,0)] \xrightarrow{\partial_0} S$$



- The differential ∂ acts on basis vectors

$$\partial(a^2b^2c) = -b \cdot a^2bc + c \cdot a^2b^2 - a \cdot ab^2c$$

Observe $\partial_0 = \left[\; a^2b \quad ac \quad bc^2 \quad b^2 \; \right]$. Thus $\mathrm{Coker}(\partial_0) = S/M$.

For $\mathbf{b} \in \mathbb{N}^n$, let $X_{\preceq \mathbf{b}}$ be the subcomplex of X consisting of all faces whose degrees are coordinatewise at most $\mathbf{b}$.

Theorem. *$\mathbb{F}_X$ is exact if and only if $X_{\preceq \mathbf{b}}$ is acyclic over $\mathbf{k}$ for all $\mathbf{b} \in \mathbb{N}^n$.*

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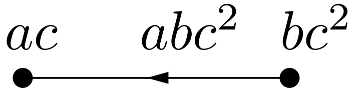
Example. Let $\mathbf{b} = (1, 1, 2)$, then $X_{\preceq \mathbf{b}}$ is the subcomplex 

- $(\mathbb{F}_X)_{\mathbf{b}}$ equals the reduced chain complex

$$\widetilde{C}(X_{\preceq \mathbf{b}}; \mathbf{k}) = 0 \to \mathbf{k} \to \mathbf{k}^2 \to \mathbf{k}$$

- $X_{\preceq \mathbf{b}}$ is contractible, so it has no reduced homology.

Matroid Ideals

- Let $\mathcal{M}$ be a matroid on the set $\{1, \ldots, n\}$, and let L be its lattice of flats.
- For every proper flat $F \in L$ let $m_x(F) = \prod_{i : i \notin F} x_i$.
- The matroid ideal M is the monomial ideal

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- A square–free monomial ideal M is a matroid ideal if and only if M is the Stanley–Reisner ideal of a matroid complex.
- A square–free monomial ideal M is a matroid ideal if and only if for every pair of monomials $m_1, m_2 \in M$ and any $i \in \{1, \ldots, n\}$ such that x_i divides both m_1 and m_2, the monomial $\operatorname{lcm}(m_1, m_2)/x_i$ is in M as well.

$$M = \langle x_1x_2, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4, x_2x_3x_5, x_4x_5 \rangle.$$

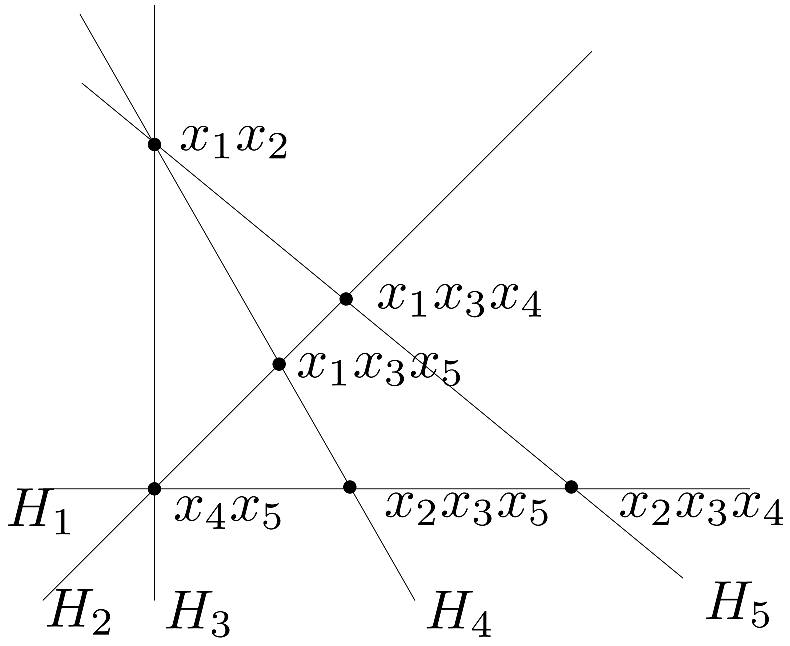
Affine Hyperplane Arrangements

- Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an affine ℓ-arrangement.
- The affine matroid $\Gamma_{\mathcal{A}}$ on $\{1, \ldots, n\}$ is the set of all subsets $S \subset [n]$ such that $S \cup \{0\}$ is a cocircuit of the matroid associated to the cone $c\mathcal{A}$.
- The affine matroid ideal $M_{\mathcal{A}}$ is generated by the monomials $m_C = x_{i_1} x_{i_2} \cdots x_{i_t}$, where $C = \{i_1, \ldots, i_t\} \in \Gamma_{\mathcal{A}}$.
- If H_0 is in general position relative to $\mathcal{A}$ ($\mathcal{A}$ is transverse to the hyperplane at infinity) then $M_{\mathcal{A}}$ is a matroid ideal.
- The ideal $M_{\mathcal{A}}$ is minimally generated by the monomials $m_x(v) = \prod_{v \notin H_i} x_i$, where v ranges over the vertices of $\mathcal{A}$.

Theorem. *Let $B_{\mathcal{A}}$ be the bounded complex of $\mathcal{A}$. Then its cellular complex $C_\bullet(B_{\mathcal{A}}, M_{\mathcal{A}})$ gives a minimal free resolution for $M_{\mathcal{A}}$.*

Affine Hyperplane Arrangements (2)

Let $\mathcal{A}$ be the 2–arrangement



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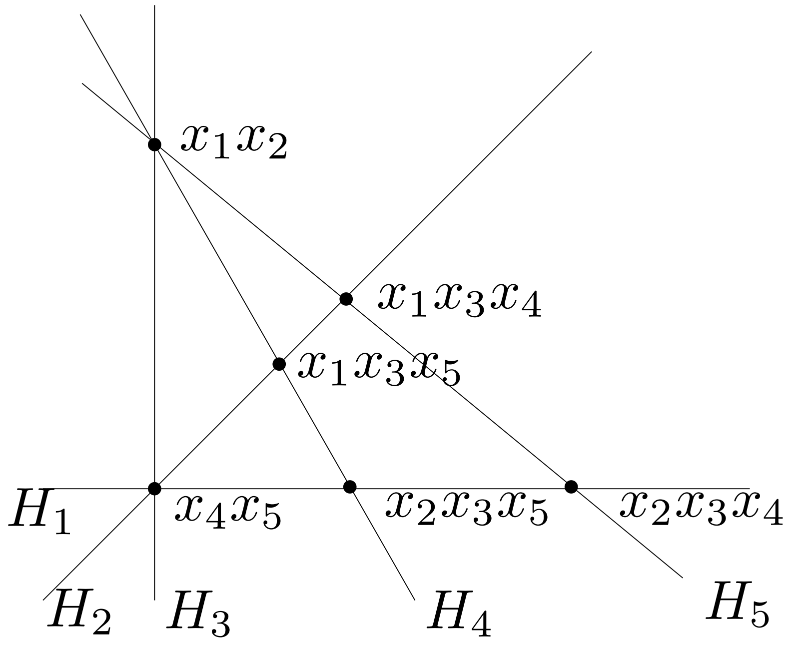

- The bounded complex $B_{\mathcal{A}}$ resolves $S/M_{\mathcal{A}}$ minimally.
- Hence $0 \longrightarrow S^4 \longrightarrow S^9 \longrightarrow S^6 \longrightarrow S$ is a minimal free resolution for $S/M_{\mathcal{A}}$.

Generic Monomial Ideals

- We say that a monomial m' strictly divides m if $\mathrm{supp}(\frac{m}{m'}) = \mathrm{supp}(m)$.
- A monomial ideal $M = \langle m_1, \ldots, m_r \rangle$ is called generic if, whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable x_s, there is a third generator m_l which strictly divides $\mathrm{lcm}(m_i, m_j)$.

Example. $M = \langle x^2y^2, x^2z^2, yz \rangle$ is generic.

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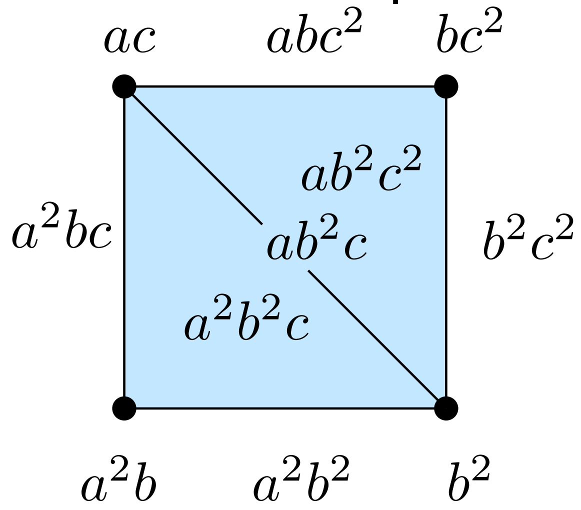
- For $\sigma \subseteq \{1, \ldots, r\}$, let $m_\sigma = \mathrm{lcm}(m_i, i \in \sigma)$.
- The Scarf complex of M consists of the following subsets:

$$\Delta_M := \{\sigma \subseteq \{1, \ldots, r\} \mid m_\sigma \neq m_\tau \, \forall \tau \subset [r], \tau \neq \sigma\}.$$

Scarf Complex

Let $M = \langle a^2b, ac, b^2, bc^2 \rangle$. The Scarf complex of M is



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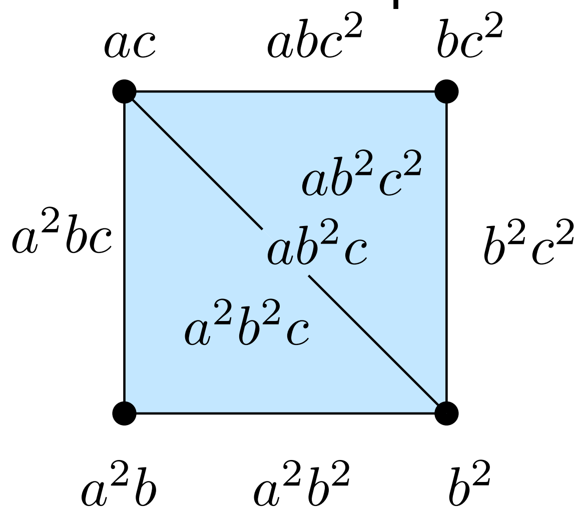

Theorem. *For M generic, the cellular complex $\mathbb{F}_{\Delta_M}$ is a minimal free resolution of* S/M.

- The minimal free resolution of S/M is

$$0 \to S^2 \to S^5 \to S^4 \to S$$

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Cohen–Macaulay Generic Deformations

- Let $M = \langle m_1, m_2, \ldots, m_r \rangle$ be a square–free monomial ideal. A deformation of M is a monomial ideal $M^* = \langle m_1^*, m_2^*, \ldots, m_r^* \rangle$, such that, for all $i \in [r]$, $\mathrm{supp}(m_i^*) = \mathrm{supp}(m_i)$.
- Denote by Δ_{M^*} the Scarf complex of a generic monomial ideal M^*. Then, $\mathbb{F}_{\Delta_{M^*}}$ is a resolution of S/M after relabeling the vertices of Δ_{M^*} with the generators of M.
- Every matroid ideal M is Cohen–Macaulay. But M^* is not necessarily Cohen–Macaulay even if M^* is a generic deformation.

Cohen–Macaulay Generic Deformations

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- Every matroid ideal M is Cohen–Macaulay. But M^* is not necessarily Cohen–Macaulay even if M^* is a generic deformation.

Theorem. *Let $M_{\mathcal{A}}$ be a matroid ideal associated to an affine ℓ–arrangement $\mathcal{A}$. Let $M_{\mathcal{A}}^*$ be a generic deformation. Then $\dim(B_{\mathcal{A}}) = \dim(\Delta_{M_{\mathcal{A}}^*})$ if and only if $M_{\mathcal{A}}^*$ is Cohen–Macaulay.*

Cohen–Macaulay Generic Matroid Ideals

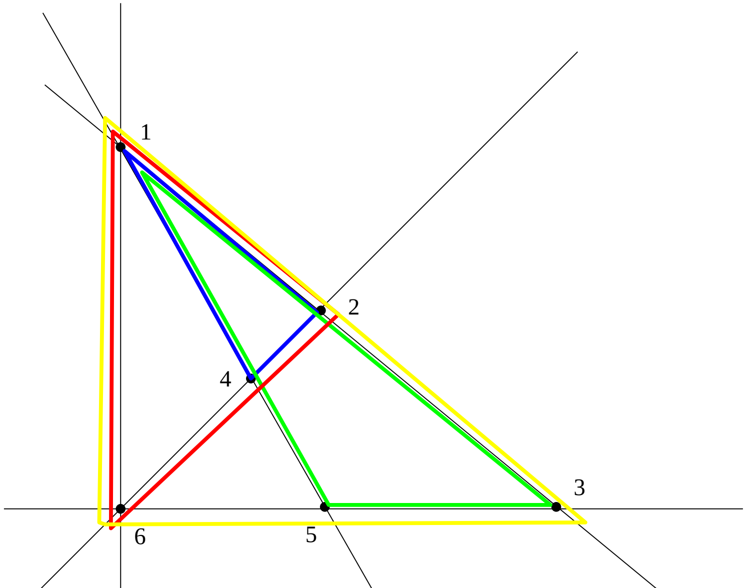
Matroid ideals always have Cohen–Macaulay generic deformations.

$$M = \langle x_1x_2, x_1x_3x_4, x_2x_3x_4, x_1x_3x_5, x_2x_3x_5, x_4x_5\rangle$$

- Fix the order $1, 2, 3, 4, 5$ in the indeterminates of S. Then

$$M^{\mathcal{A}} = \langle x_1x_2, x_1^2x_3^2x_4^2, x_2^2x_3^2x_4^2, x_1^3x_3^3x_5^3, x_2^3x_3^3x_5^3, x_4^3x_5^3\rangle$$

  is a **CM generic deformation** of M.
- $\Delta_{M^{\mathcal{A}}}$ has facets $\{1, 2, 4\}, \{1, 3, 5\}, \{1, 2, 6\}, \{1, 3, 6\}$

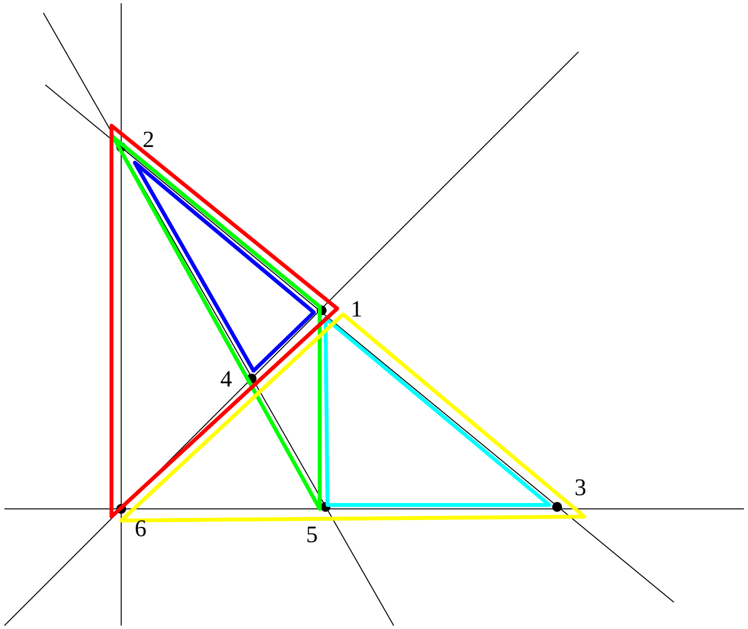


Non-minimal resolution

- Fix the order $1, 3, 4, 2, 5$. Then

$$M^{\mathcal{A}} = \langle x_1x_3x_4, x_1^2x_2^2, x_2^2x_3^2x_4^2, x_1^3x_3^3x_5^3, x_2^3x_3^3x_5^3, x_4^3x_5^3 \rangle$$

  is the **CM generic deformation** of M associated to the given order.
- $\Delta_{M^{\mathcal{A}}}$ has facets $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 6\}$



- $\mathbb{F}_{\Delta_{M^{\mathcal{A}}}}$ is equal to $0 \longrightarrow S^5 \longrightarrow S^{10} \longrightarrow S^6 \longrightarrow S$

Theorem. *Let* $\mathcal{A}$ *be an affi ne* ℓ*–arrangement transverse to the hyperplane at infi nity. Then, there exists a Cohen–Macaulay generic deformation of the matroid ideal* $M_{\mathcal{A}}$ *that gives a minimal free resolution of* $S/M_{\mathcal{A}}$ *if (and only if) the arrangement* $\mathcal{A}$ *is supersolvable.*