Classification of Finite Dynamical Systems

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  - Can we find a more efficient simulation of a given system?

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C. L. Barrett, H. S. Mortveit, C. M. Reidys at LANL

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- A set of entities having state values
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- A dependency graph in which the entities interact
- An update schedule which specifies how the local functions are to be composed to generate a global update function.

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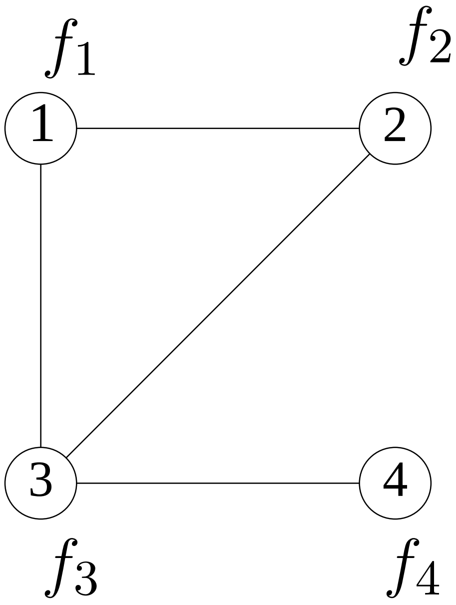
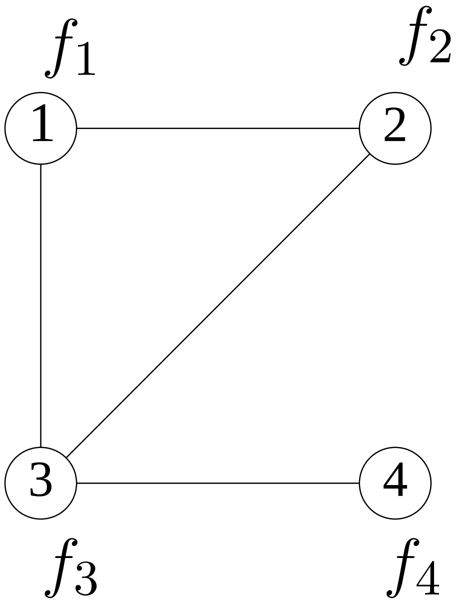
SDS example



SDS example

- Four binary entities



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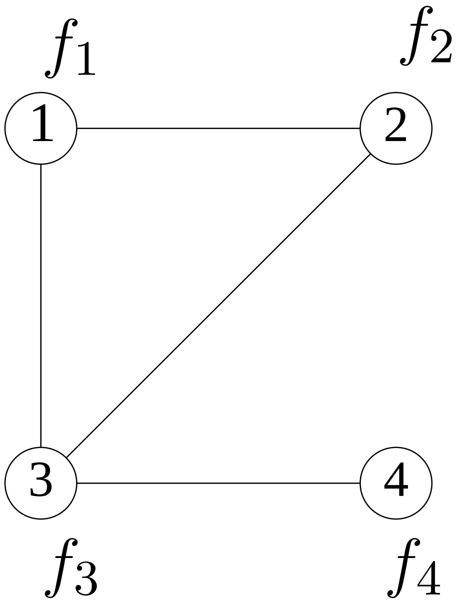

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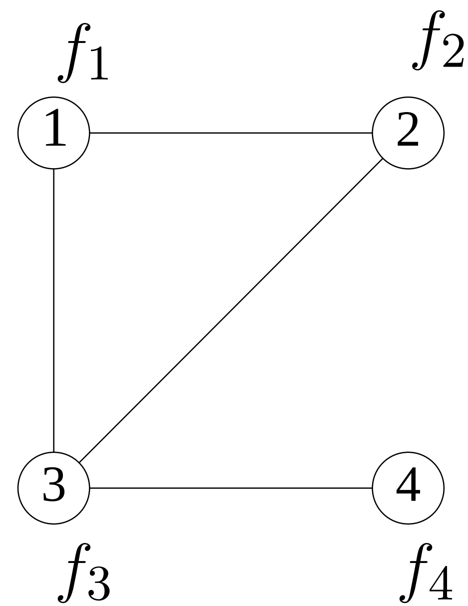

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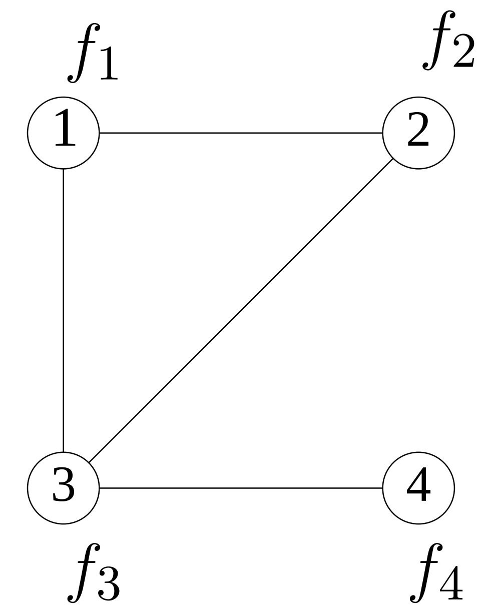

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- Compute $F(0, 0, 0, 0)$

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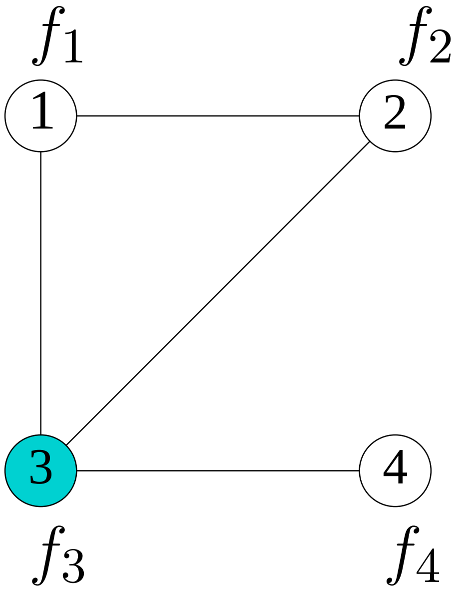

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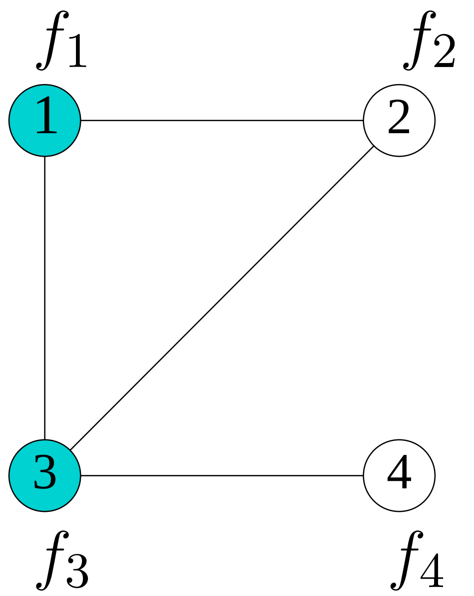

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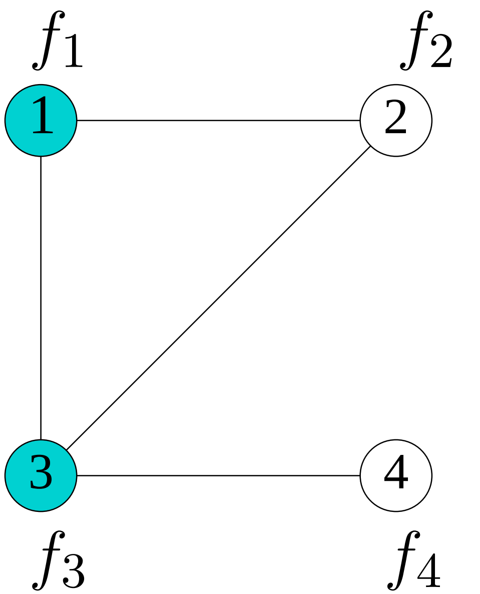

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  - Combinatorial and algebraic aspects of SDS.
    - A sharp upper bound on the number of different SDS obtained by rescheduling: the number of acyclic orientations of the underlying graph.
    - An upper bound on the number of non-dynamically equivalent SDS, that is, SDS with non-isomorphic state spaces.

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- The update schedule is a permutation on the number of entities. No capability of updating a local function more than once.
- SDS assumes a fixed underlying graph. In applications the dependency graph frequently **varies over time**.

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$$f^i(x) = (x_1, \ldots, x_{i-1}, f_i^i(x), x_{i+1}, \ldots, x_n)$$

R. Laubenbacher, B. Pareigis

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- A finite dynamical system is an element of the set $\mathcal{S}$ of n-tuples of functions

$$\mathcal{S} = \{(f^1, \ldots, f^n) \mid f^i \in L_n^i\}$$

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Galois Correspondence

Theorem. *There is a Galois correspondence between the power set of $\mathcal{S}$ and the set $\mathcal{G}$ of subgraphs of the complete graph K_n on the vertex set $\{1, \ldots, n\}$*

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The graph $\Phi(f)$ encodes the dependency relations among the local functions f^i

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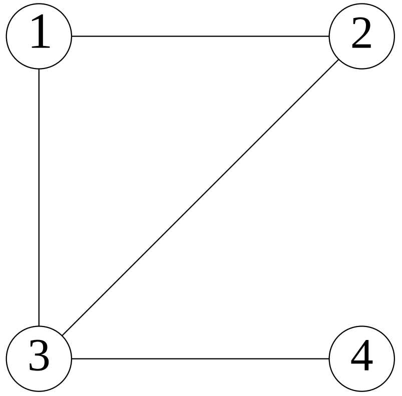

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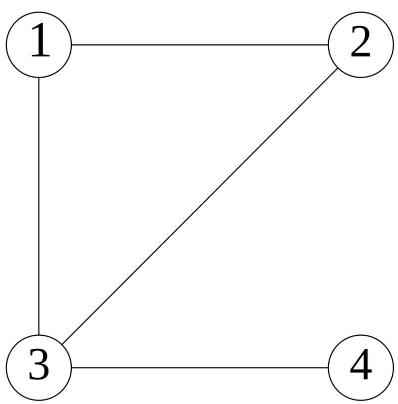

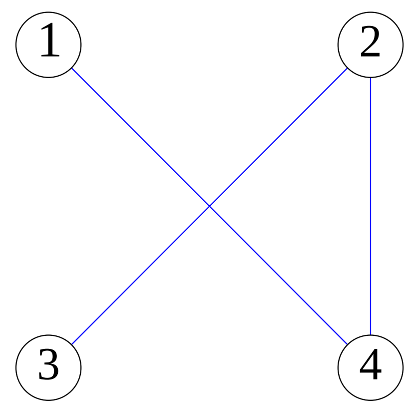

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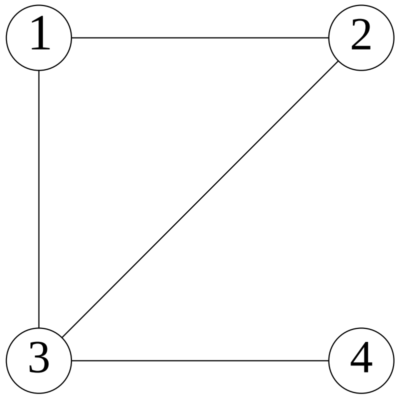

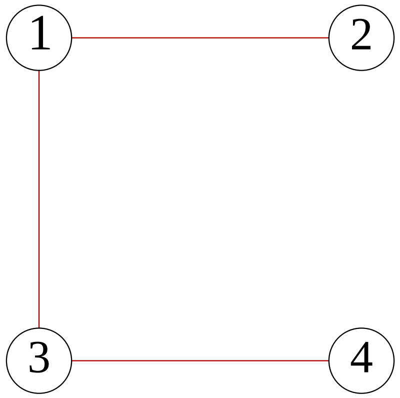

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- A linear system $f = (f^1, \ldots, f^n)$ can be represented by an $(n \times n)$-matrix with entries in $\mathbb{K}$.
- *P. Cull* (1970) represented a Switching Net with a $2^n \times 2^n$-matrix A, the function matrix, that has as its rows the products of the n local functions.
  - The characteristic polynomial of A has the form $x^k(x^{r_1} + 1) \cdots (x^{r_s}+1)$, where k is the number of transient states and the r's are the lenghts of the various cycles.
  - Behavior of a net obtained from the eigenvectors.

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- A system $f = (f^1, \ldots, f^n) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ is called linear if all functions f_i^i are $\mathbb{K}$-linear polynomials.

Lemma. *Let G be a graph on n vertices. Then there exists a linear system $l_G = (l^1, \ldots, l^n) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ such that $\Phi(l_G) = G$.*

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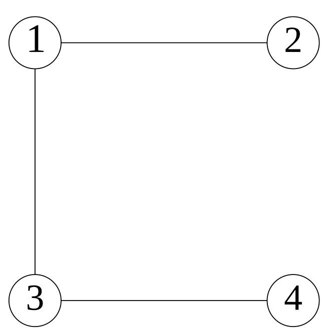

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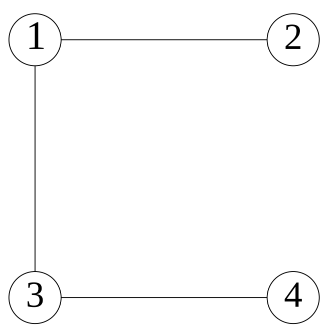

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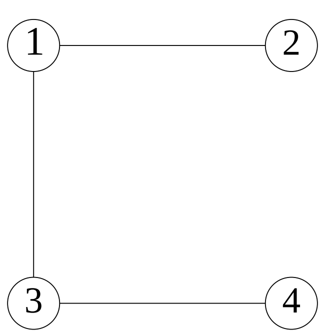

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$$l_1^1 = x_4$$
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Theorem. *Let $f = (f^1, \ldots, f^n) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ be a system. Then there exists a linear system $l_G = (l^1, \ldots, l^n) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ such that $\Phi(l_G) = \Phi(f)$.*

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The linear system l_G corresponding to the adjacency matrix of the complement of G is called the linearization of the system f.

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Graph Equivalence

We say that $(f^1, \ldots, f^n), (g^1, \ldots, g^n) \in \mathcal{S}$ are graph equivalent if and only if $\Phi((f^1 \ldots, f^n))$ is isomorphic to $\Phi((g^1, \ldots, g^n))$.

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Let S_n be the group of permutations on n elements. For any $(n \times n)$-matrix M and $\pi \in S_n$, πM is the $(n \times n)$-matrix such that $(\pi M)_{ij} = M_{\pi^{-1}(i)\pi^{-1}(j)}$.

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Theorem. *Let f and g be two systems on $\mathbb{K}^n$. f is graph equivalent to g, that is, $\pi(\Phi(f)) = \Phi(g)$ for some $\pi \in S_n$ if and only if $\pi \cdot M_{l_{\Phi(f)}} = M_{l_{\Phi(g)}}$.*

Upper Bound for Sequential Systems

- Let W_t be the set of all words on $\{1, \ldots, n\}$ of length t.
- Let $f \in \mathcal{S}$, and $\pi = (i_1, \ldots, i_t) \in W_t$. Denote by f^π the finite dynamical system given by

$$f^{i_t} \circ \cdots \circ f^{i_1}.$$

- Let $H_\pi(f)$ be the graph on t vertices, corresponding to $i_1, \ldots, i_t$. Then (v_a, v_b) is an edge in $H_\pi(f)$ if and only if
  - $i_a \neq i_b$ and,
  - the edge (i_a, i_b) is not in $\Phi(f)$.

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Upper Bound Theorem

Theorem. *Let* $f = (f^1, \ldots, f^n)$ *be a system of local functions on* $\mathbb{K}^n$*, and let* $F_{W_t}(f) = \{f^\pi \mid \pi \in W_t\}$*. Then*

$$|F_{W_t}| \leq |\{\mathrm{Acyc}(H_\pi) \mid \pi \in W_t\}| = \sum_{\pi \in W_t} |\{\mathrm{Acyc}(H_\pi)\}|.$$

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Observation. *If* $\pi \in S_n$*, then* $\{\mathrm{Acyc}(H_\pi) : \pi \in W_n\} = \{\mathrm{Acyc}(\overline{\Phi(f)})\}$*.*

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Upper Bound Theorem

Theorem. *Let* $f = (f^1, \dots, f^n)$ *be a system of local functions on* $\mathbb{K}^n$*, and let* $F_{W_t}(f) = \{f^\pi \mid \pi \in W_t\}$*. Then*

$$|F_{W_t}| \le |\{\mathrm{Acyc}(H_\pi) \mid \pi \in W_t\}| = \sum_{\pi \in W_t} |\{\mathrm{Acyc}(H_\pi)\}|.$$

Observation. *If* $\pi \in S_n$*, then* $\{\mathrm{Acyc}(H_\pi) : \pi \in W_n\} = \{\mathrm{Acyc}(\overline{\Phi(f)})\}$*.*

Thus the upper bound for the number of different SDS is recovered. This bound is known to be sharp.

Dynamically Equivalent Systems

- Two maps $F, G : \mathbb{K}^n \to \mathbb{K}^n$ are dynamically equivalent if there exists a bijection $\varphi : \mathbb{K}^n \to \mathbb{K}^n$ such that
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- An upper bound for dynamically non-equivalent SDS is known (Reidys). This upper bound relies on the fact that conjugacy yields an **SDS** with the same graph and local functions.

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$$G = \varphi \circ F \circ \varphi^{-1}$$
- Let $f = (0, x_3, x_2) : \mathbb{K}^3 \to \mathbb{K}^3$.
- $\Phi(f)$ is the graph on three vertices $1, 2, 3$ with edges $(1, 2), (1, 3)$.
- Exactly two functionally non-equivalent systems $f^3 \circ f^2 \circ f^1$ and $f^1 \circ f^2 \circ f^3$.

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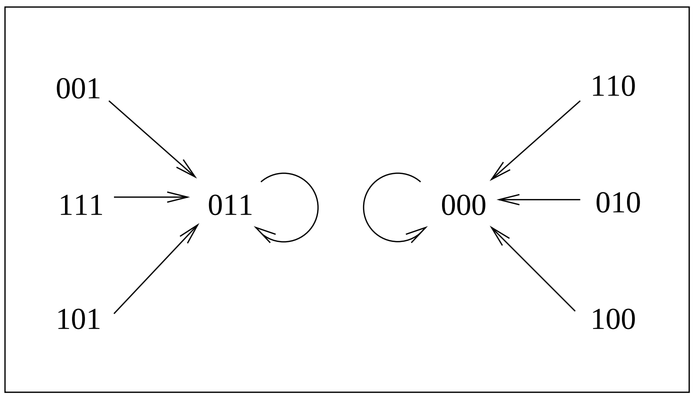
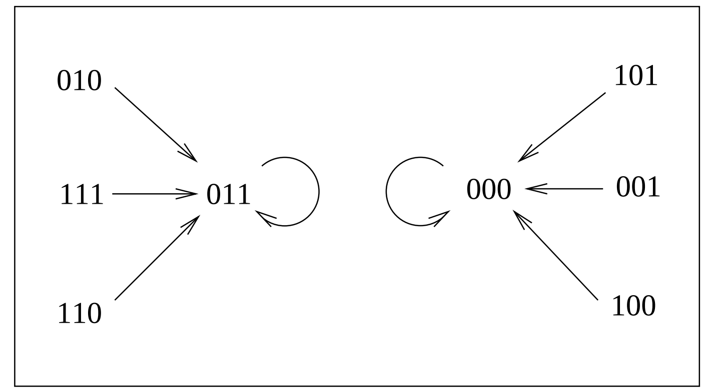
Dynamically Equivalent Systems (2)

- The state spaces of f^{id} and $f^1 \circ f^2 \circ f^3$





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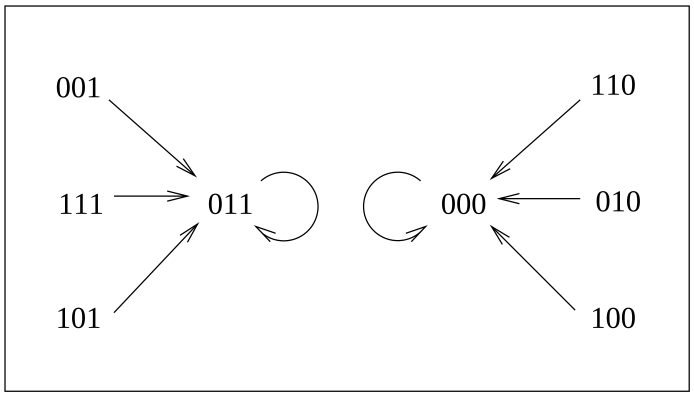

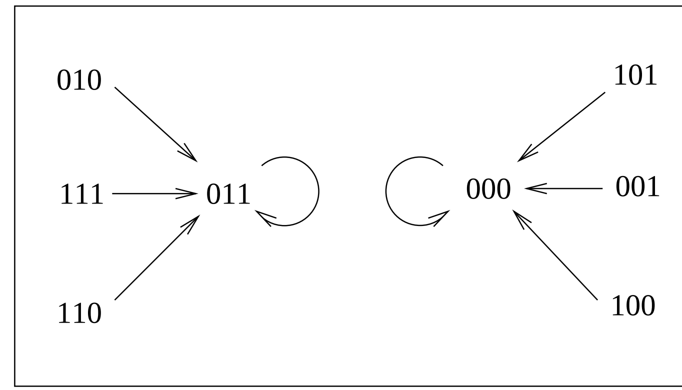

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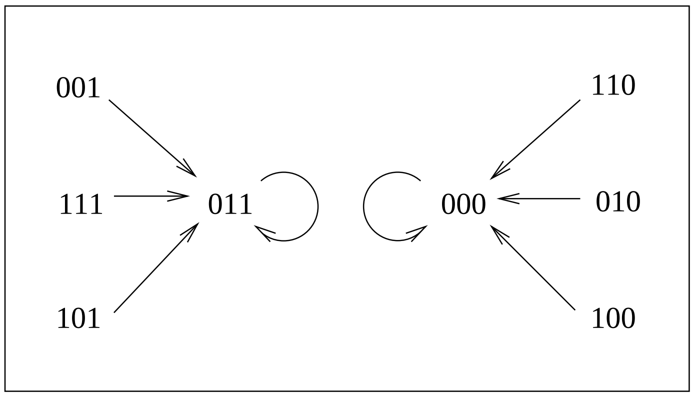

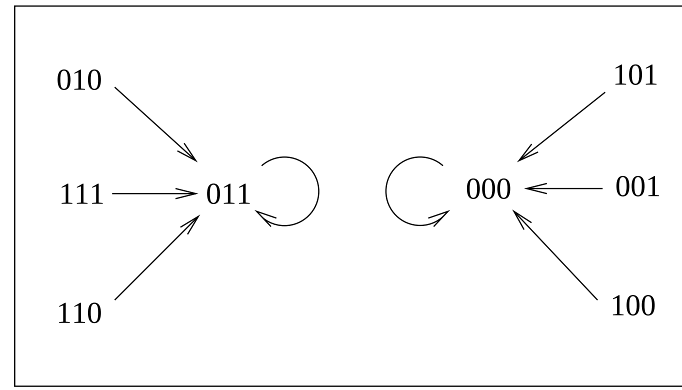

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