Experimentation at the Frontiers of Reality in Schubert Calculus

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Schubert calculus

Schubert problem: Given two lists of

1. Reference flags, and
2. Schubert conditions
   (that specify how a k-plane meets a reference flag),

ask how many k-planes meet the conditions relative to the given flags.

Schubert calculus: Combinatorially determine the number of solutions when the reference flags are general, over $\mathbb{C}$.

Example: How many lines meet four given general lines in $\mathbb{P}^3$?

What about **real** solutions when the reference flags are real?

Shapiro conjecture

In 1994, Boris and Michael Shapiro conjectured:

Shapiro Conjecture

*Geometric problems in Schubert calculus involving **tangent flags** have only real solutions.*

Originally considered too strong to be true, but became well-known & studied due to overwhelming experimental/computational evidence, conducted on **Grassmannians**.

Sottile, Real Schubert Calculus, Exp. Math., 9, (2000), 161–182.

This is a theorem for Schubert problems on Grassmanians.

Mukhin, Tarasov and Varchenko, The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz, Annals of Mathematics.

MONOTONE CONJECTURE

The Shapiro Conjecture is false for flag varieties–computer experimentation revealed a shockingly simple counterexample. Further experimentation suggested a refinement, the **Monotone Conjecture**.

Sivan, et. al, Experimentation and conjectures in the real Schubert calculus for flag manifolds, Exp. Math., 15, (2006), 199–221.

This experiment occupied 15.76 GHz-years of computing, solving 520,420,135 polynomial systems, representing 1126 Schubert problems on 29 different flag manifolds.

Secant conjecture

Eremenko–Gabrielov–M. Shapiro–Vainshtein proved a special case of the **Monotone Conjecture**. In terms of Schubert calculus on the Grassmannian, this leads to a new **Secant Conjecture** involving **disjoint secant flags**. The Shapiro Conjecture is a limiting case of this Secant Conjecture.

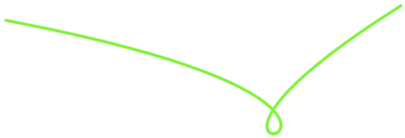
Secant Conjecture

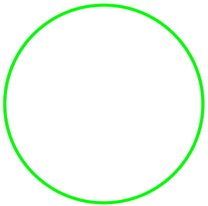
Fix a real rational normal curve C.
If a Schubert problem is given by real reference flags which are ***secant along disjoint intervals*** *on C, then the Schubert problem has all solutions real.*

The four secant line Schubert problem: Given four secant lines to a real twisted cubic, how many real lines meet all four?

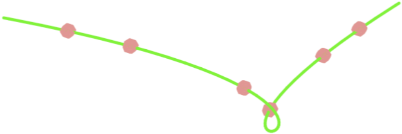
The four secant line Schubert problem

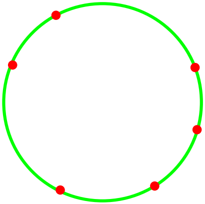
Begin with a rational normal curve in 3-space:



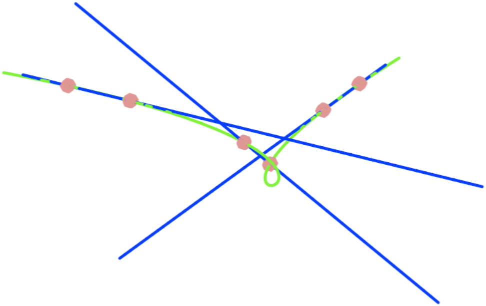


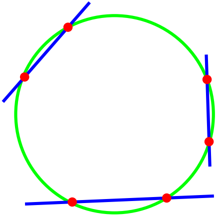
Select three pairs of points on the rational normal curve:



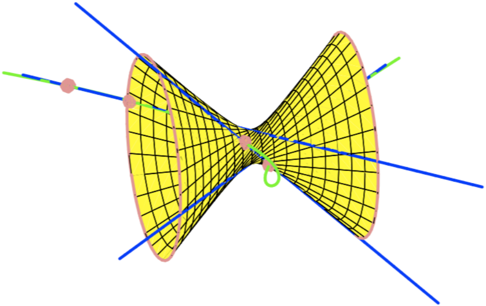


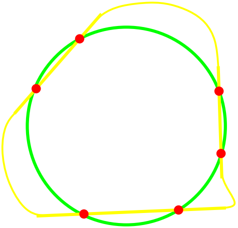
Consider the secant lines through those points:



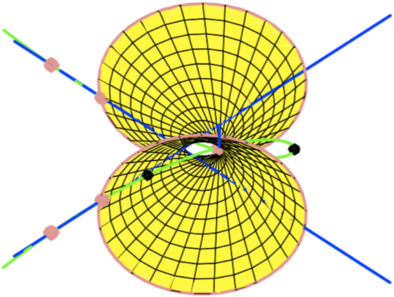


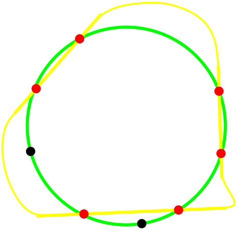
Introduce the hyperboloid of one sheet containing the three secant lines:



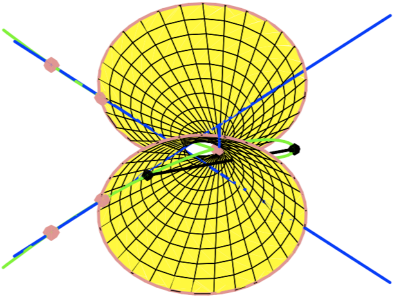


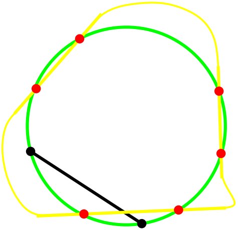
Add one more pair of points on the rational normal curve, in adjacent segments of the curve:



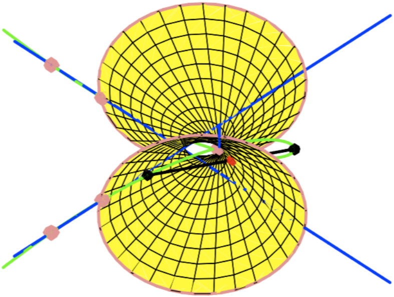


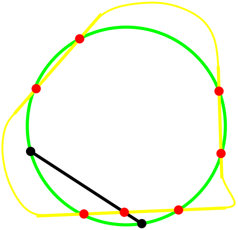
Since one point is inside the hyperboloid and one is outside, the line segment joining them crosses the hyperboloid:





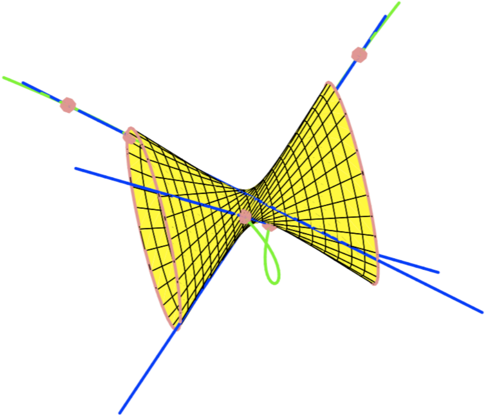
So there is one (hence two) real solution(s).

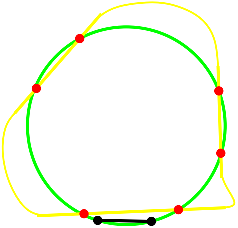




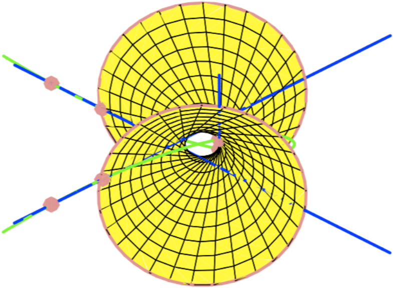
The four secant line Schubert problem

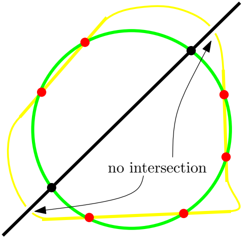
Two points in the same segment may lead to no real solutions, if that segment leads to non-disjoint intervals:



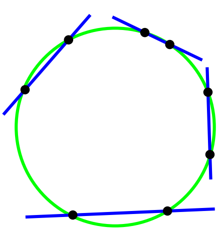


This view shows the same behavior “inside” the hyperboloid:

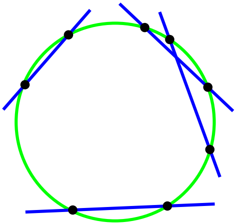




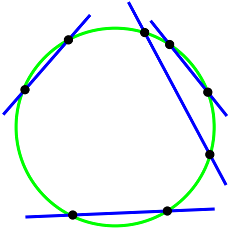
Secant conjecture for the four line problem:



2 real roots



2 real roots



0 or 2 real roots

Description of computational experiment

Goal: Test as many instances of the conjecture as possible, in particular, all Schubert problems on all small Grassmannians.

- Our team wrote software to study this Secant Conjecture on personal computers and a 1.1 teraflop Beowulf cluster whose day job is Calculus instruction.
- The automated computation is organized from a MySQL database.
- Processors query the database for problems to solve, record what is computing, and update the results after successful computation.
- The computation is robust–it automatically recovers from failures, and is repeatable using fixed random seeds and a standard pseudorandom number generator.
- Its progress is monitored from the web and fine-tuned with MySQL browsers and other software tools we have written.

General purpose software:

- **Perl** used to manage the high-performance computations in the Beowulf cluster (192 computers with 2 or 4 CPUs) and store the results in a MySQL database.
- **PHP** used to mine the database and display its contents on a web page.

Mathematical software:

- **Singular/Macaulay2/Fermat** used to represent Schubert problems as systems of polynomial equations and compute *eliminants*.
- **Maple/Sage/Maxima (SARAG)** used to find the number of real roots of the eliminant.

Current status of experiment

To date, we have

- used over 467 GHz-years of computation.
- solved over 1 billion polynomial systems arising from 457 Schubert problems.

Much more is planned.

This includes using a second supercomputer to verify our results.

`http://www.math.tamu.edu/~secant`