The Control Polyhedron of a Rational Bézier Surface

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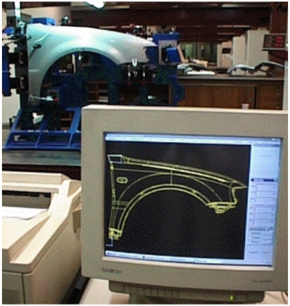
Algebraic geometry applications to geometric modeling

Geometric modeling uses polynomials to build computer models for industrial design and manufacture.

Algebraic geometry investigates the algebraic and geometric properties of polynomials.



Bézier Curves

Bézier curves are **parametric curves** used in computer graphics to model smooth curves. Fundamental objects in geometric modeling.

- First introduced by Charles Hermite and Sergei Bernstein.
- Widely publicized in the 1960's by Pierre Bézier (Renault), and Paul De Casteljau (Citroën) in the design of automobile bodies.
- Used in animation software such as Adobe Flash to outline movement.
- Used also in the design of fonts:
  - Quadratic Bézier curves are used in True Type fonts,
  - cubic Bézier curves are used in Type 1 fonts,
  - cubic Bézier curves are also used in the TEX fonts.

Bézier Curves

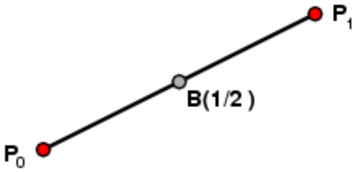
$$\mathbf{B(x)} := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i} \, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$B(x) := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$B(x) = (1-x)\mathbf{P}_0 + x\mathbf{P}_1$$



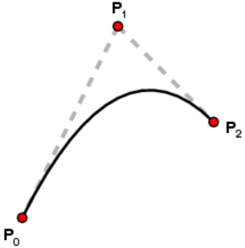
LINEAR PRECISION

$$\sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i} \frac{i}{d} = x$$

$$\mathbf{B(x)} := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$B(x) = (1-x)^2\mathbf{P}_0 + 2x(1-x)\mathbf{P}_1 + x^2\mathbf{P}_2$$



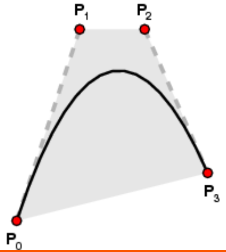
Endpoint interpolation

$$B(0) = \mathbf{P}_0, \quad B(1) = \mathbf{P}_2$$

$$\mathbf{B(x)} := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

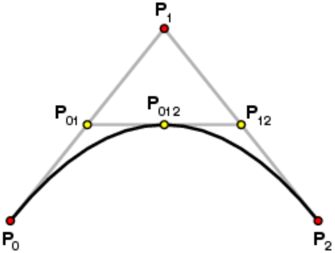
where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

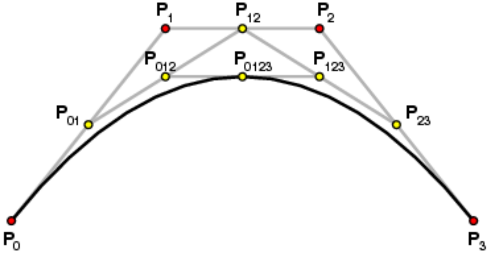
$B(x) = (1-x)^3\mathbf{P}_0 + 3x(1-x)^2\mathbf{P}_1 + 3x^2(1-x)\mathbf{P}_2 + x^3\mathbf{P}_3$



Convex hull

The curve $B([0,1])$ is contained in the convex hull of the control points.


P_1
P_{01}
P_{012}
P_{12}
P_0
P_2

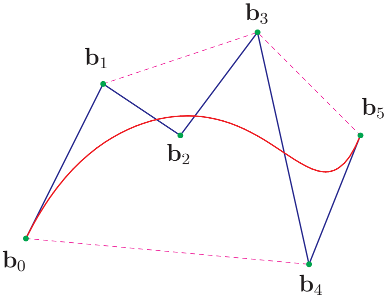

P_1
P_{12}
P_2
P_{012}
P_{0123}
P_{123}
P_{01}
P_{23}
P_0
P_3

Control Polygons

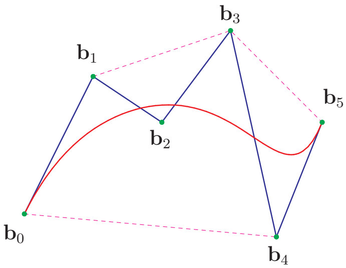
Let $B(x)$ be the Bézier curve given by

$$B(x) = \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i} \mathbf{b}_i, \qquad \text{with } x \in [0,1],$$

with $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_d$ control points in $\mathbb{R}^n$. The corresponding **control polygon** is the union of the line segments $\overline{\mathbf{b}_0, \mathbf{b}_1}, \overline{\mathbf{b}_1, \mathbf{b}_2}, \ldots, \overline{\mathbf{b}_{d-1}, \mathbf{b}_d}$.



$$B(x) = (1-x)^5\mathbf{b}_0 + 5x(1-x)^4\mathbf{b}_1 + 10x^2(1-x)^3\mathbf{b}_2 + 10x^3(1-x)^2\mathbf{b}_3 + 5x^4(1-x)\mathbf{b}_4 + x^5\mathbf{b}_5.$$



The number of points in which a Bézier curve meets a line is bounded by number of points in which its control polygon meets the same line.

Generalizing this property to surfaces is similar to the open problem of finding a satisfactory multivariate generalization of Descartes' rule of signs.

Rational Bézier curves add adjustable weights to provide closer approximations to arbitrary shapes.

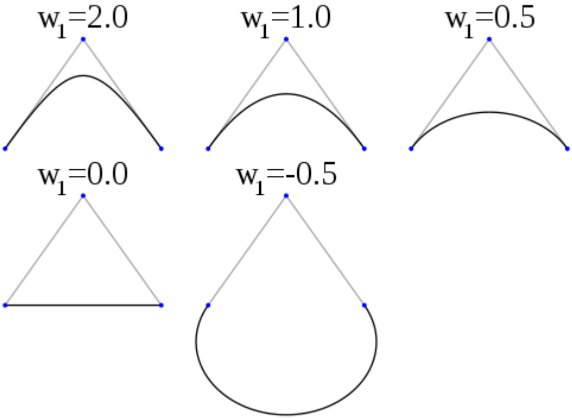
BERNSTEIN POLYNOMIALS

$$\beta_{\mathbf{i;d}}(\mathbf{x}) := \binom{d}{i} x^i (1-x)^{d-i}$$

Given **weights** $w_0, w_1, \ldots, w_d$ in $\mathbb{R}_>$ and control points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_d$ in $\mathbb{R}^n$, the **rational Bézier curve** is

$$\mathbf{B}(\mathbf{x}) := \frac{\sum_{i=0}^d w_i \beta_{i;d}(x)\, \mathbf{b}_i}{\sum_{i=0}^d w_i \beta_{i;d}(x)} \; : \; [0,1] \longrightarrow \mathbb{R}^n .$$

$$B(x) = \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(x)\, \mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)}$$



Toric Bézier Curves

- For each $i = 0, \ldots, d$ redefine the **Bernstein polynomial** $\beta_{i;d}(x)$,

$$\beta_{\mathbf{i;d}}(\mathbf{x}) \;:=\; x^i(d-x)^{d-i}\,.$$

  Substituting $x = dy$ and multiplying by $\binom{d}{i}d^{-d}$ for normalization, this becomes the usual Bernstein polynomial.

- Given weights $w_0, \ldots, w_d \in \mathbb{R}_>$ and control points $\mathbf{b}_0, \ldots, \mathbf{b}_d \in \mathbb{R}^n$ $(n = 2, 3)$, the parametrized **toric Bézier curve** is defined by

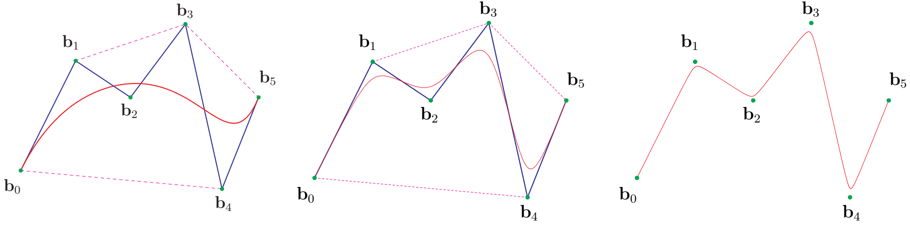
$$\mathbf{B}(\mathbf{x}) \;:=\; \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(x)\,\mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} \;:\; [0, d] \;\longrightarrow\; \mathbb{R}^n\,.$$

- Differ from the rational Bézier curves in that the degree is encoded by the domain. This **linear reparametrization** does not affect the resulting curve.

Toric Bézier Curve Deformations

Theorem (Craciun-G-Sottile)

Given control points in $\mathbb{R}^n$ and $\epsilon > 0$, there is a choice of weights so that the toric Bézier curve lies within a distance ϵ of the control polygon.



Factorization of the Toric Bézier Curve Map

Let $\triangle^d \subset \mathbb{R}^{d+1}$ be the **standard simplex** of dimension d with homogeneous coordinates

$$[z_0, z_1, \ldots, z_d] \;:=\; \frac{1}{\sum_{i=0}^{d} z_i}(z_0, z_1, \ldots, z_d)\,.$$

The map $B(x) = \dfrac{\sum_{i=0}^{d} w_i \beta_{i;d}(x)\, \mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} \;:\; [0,d] \to \mathbb{R}^n$ admits the factorization:

$$B(x) \,:\, [0,d] \xrightarrow{\beta} \triangle^d \xrightarrow{w\cdot} \triangle^d \xrightarrow{\pi} \mathbb{R}^n\,, \text{ where}$$

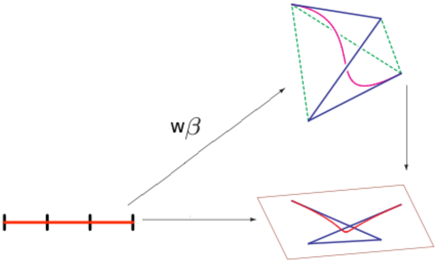
$$\beta : [0,d] \to \triangle^d, \quad x \longmapsto [\beta_{0;d}(x), \beta_{1;d}(x), \ldots, \beta_{d;d}(x)].$$

$$\mathbf{w}\cdot : \triangle^d \to \triangle^d, \quad [z_0, z_1, \ldots, z_d] \longmapsto [w_0 z_0, w_1 z_1, \ldots, w_d z_d]\,.$$

$$\pi : \mathbb{R}^{d+1} \to \mathbb{R}^n, \quad (z_0, \ldots, z_d) \longmapsto \sum_{i=0}^{d} z_i\, \mathbf{b}_i\,.$$

Factorization of the Toric Bézier Curve Map

$$B(x):\ [0,d] \xrightarrow{\beta} \Delta^d \xrightarrow{w\cdot} \Delta^d \xrightarrow{\pi} \mathbb{R}^n,$$



$X = \beta([0,d])$ is the **positive real part** of the **rational normal curve**.

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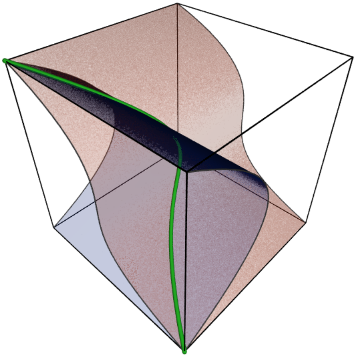
The **(affine) rational normal curve** is the image of $x \longmapsto (x,\ x^2,\ \ldots,\ x^d)$.



When $d = 3$, this curve is called the **twisted cubic**.

Defined **parametrically** by

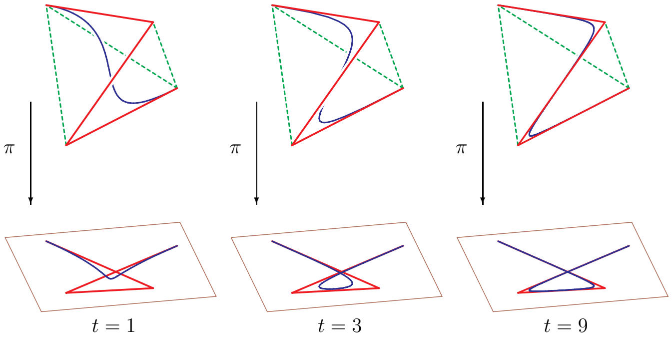
$$x \longmapsto (x,\ x^2,\ x^3),$$

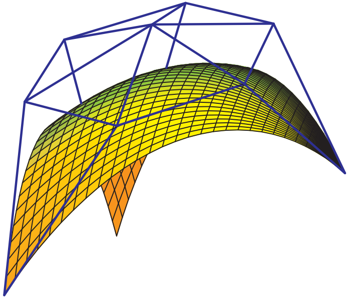
and **implicitly** by the equations

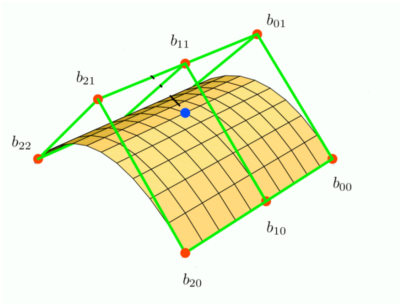
$$Y - X^2 = 0,\ Z - X^3 = 0.$$

THEOREM (CRACIUN-G-SOTTILE)

Given control points in $\mathbb{R}^n$ and $\epsilon > 0$, there is a choice of weights so that the toric Bézier curve lies within a distance ϵ of the control polygon.



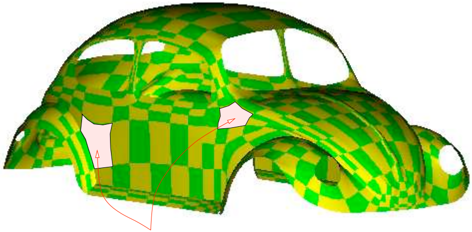


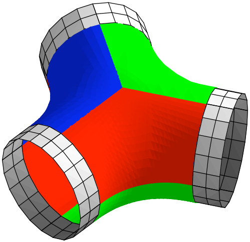


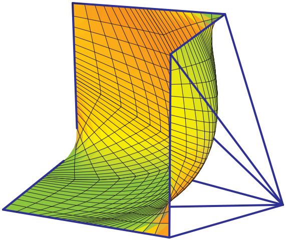
rational Bézier triangular patch

rational Bézier rectangular patch

Multi-sided Patches







Toric Bézier Surface Patches of Shape Δ

A **polygon** $\Delta \subset \mathbb{R}^2$ with integer vertices is given by **side inequalities**

$$\Delta = \left\{(x,y) \in \mathbb{R}^2 \mid h_s(x,y) := b_s x + c_s y + d_s \geq 0, \;\; \text{for each side } s \text{ of } \Delta\right\},$$

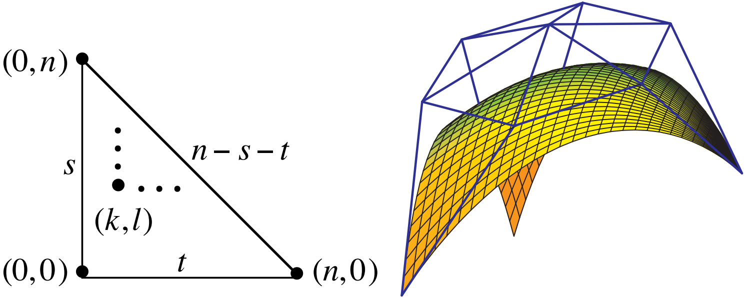
where (b_s, c_s) is an inward pointing primitive normal vector.

For each $\mathbf{a} \in \mathscr{A} := \Delta \cap \mathbb{Z}^2$, define a **toric Bézier function**

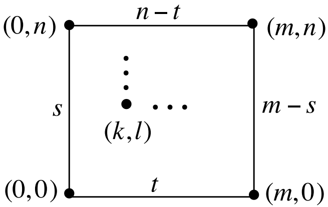
$$\beta_{\mathbf{a},\mathscr{A}}(\mathbf{x},\mathbf{y}) := \prod_{s \text{ side of } \Delta} h_s(x,y)^{h_s(\mathbf{a})} : \Delta \longrightarrow \mathbb{R}.$$

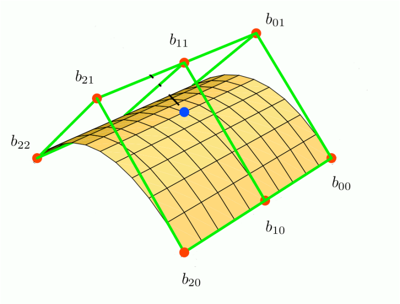
Given positive weights $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}_>$ and control points $\mathscr{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^n$, the **toric Bézier surface of shape Δ** is parametrized by

$$F_{\mathscr{A},w,\mathscr{B}}(\mathbf{x}) \;:=\; \frac{\sum_{\mathbf{a}\in\mathscr{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathscr{A}}(\mathbf{x})\, \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathscr{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathscr{A}}(\mathbf{x})} : \Delta \longrightarrow \mathbb{R}^n.$$



$$F(s,t) = \sum_{k,l} \frac{\binom{n}{k,l} s^k t^l (n-s-t)^{n-k-l}}{n^n} \mathbf{b}_{kl}$$





$$F(s,t) = \sum_{k,l} \frac{\binom{m}{k}\binom{n}{l} s^k (m-s)^{m-k} t^l (n-t)^{n-l}}{m^m n^n} \mathbf{b}_{kl}$$

Factorization of the Toric Bézier Surface Map

Let $\triangle^{\mathscr{A}} \subset \mathbb{R}^{\mathscr{A}}$ is the standard simplex of dimension $|\mathscr{A}|-1$ with homogeneous coordinates

$$[z_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}] \ := \ \frac{1}{\sum_{\mathbf{a}\in\mathscr{A}} z_{\mathbf{a}}} (z_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}),$$

The map $F(\mathbf{x}) : \Delta_{\mathscr{A}} \longrightarrow \mathbb{R}^n$ admits the following **factorization**:

$$F(\mathbf{x}) \ : \ \Delta_{\mathscr{A}} \xrightarrow{\beta_{\mathscr{A}}} \triangle^{\mathscr{A}} \xrightarrow{w\cdot} \triangle^{\mathscr{A}} \xrightarrow{\pi_{\mathscr{B}}} \mathbb{R}^n,$$

$$\beta_{\mathscr{A}}(\mathbf{x}) \ := \ [\beta_{\mathbf{a},\mathscr{A}}(\mathbf{x}) \mid \mathbf{a} \in \mathscr{A}] : \Delta_{\mathscr{A}} \to \triangle^{\mathscr{A}},$$

$$\mathbf{w}\cdot[z_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}] \ := \ [w_{\mathbf{a}} z_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}] : \triangle^{\mathscr{A}} \to \triangle^{\mathscr{A}}.$$

$$\pi_{\mathscr{B}}(\mathbf{z}) \ := \ \sum_{\mathbf{a}\in\mathscr{A}} z_{\mathbf{a}}\, \mathbf{b}_{\mathbf{a}} : \mathbb{R}^{\mathscr{A}} \to \mathbb{R}^n.$$

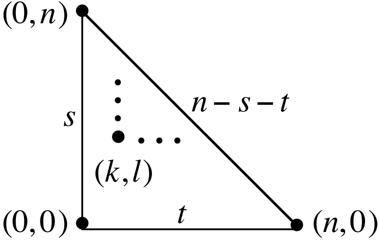
The map $F(\mathbf{x}) : \Delta_{\mathscr{A}} \longrightarrow \mathbb{R}^n$ admits the following **factorization**:

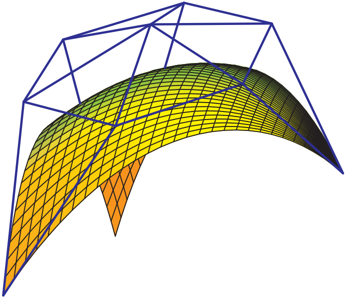
$$F(\mathbf{x}) : \ \Delta_{\mathscr{A}} \xrightarrow{\beta_{\mathscr{A}}} \triangle^{\mathscr{A}} \xrightarrow{w\cdot} \triangle^{\mathscr{A}} \xrightarrow{\pi_{\mathscr{B}}} \mathbb{R}^n,$$

The image $\mathbf{X}_{\mathscr{A}} := \beta_{\mathscr{A}}(\Delta_{\mathscr{A}}) \subset \triangle^{\mathscr{A}}$ is the **positive part of the toric variety associated to the polygon** $\Delta_{\mathscr{A}}$

Acting on $X_{\mathscr{A}}$ by the map $w\cdot$ gives a **translated toric variety** $X_{\mathscr{A},w}$

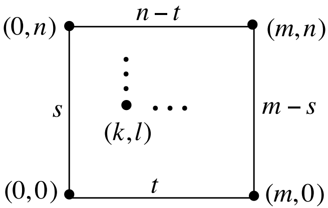
We call $X_{\mathscr{A},w}$ a **lift** of the toric Bézier patch $\mathbf{Y}_{\mathscr{A},\mathbf{w},\mathscr{B}} := \pi_{\mathscr{B}}(X_{\mathscr{A},w})$

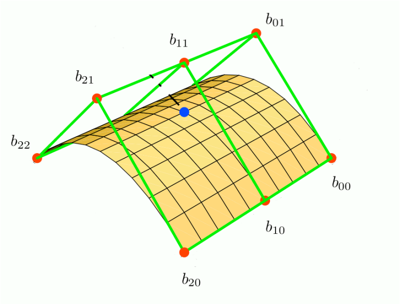




$$F(s,t) = \sum_{k,l} \frac{\binom{n}{k,l} s^k t^l (n-s-t)^{n-k-l}}{n^n} \mathbf{b}_{kl}$$

The corresponding toric variety is a Veronese surface of degree n.





$$F(s,t) = \sum_{k,l} \frac{\binom{m}{k}\binom{n}{l} s^k (m-s)^{m-k} t^l (n-t)^{n-l}}{m^m n^n} \mathbf{b}_{kl}$$

The corresponding toric variety is a is the Segre product of two rational normal curves of degrees n and m.

What is the Significance of the Control Net?

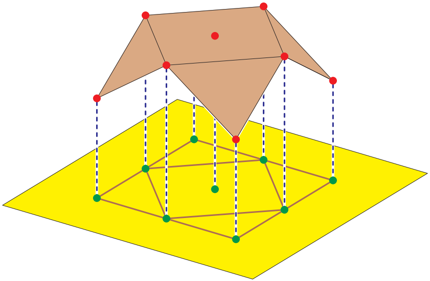
These **control nets** encode certain C^0 spline surfaces called **regular control surfaces**. While not unique, regular control surfaces are exactly the possible limiting positions of a Bézier patch when the weights are allowed to vary.

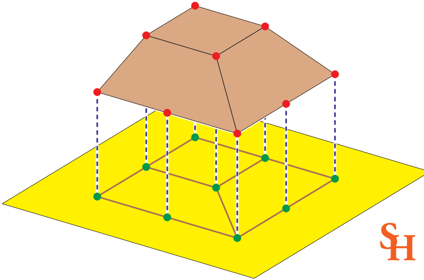
Let $\mathscr{A} \subset \mathbb{R}^2$ and $\lambda \colon \mathscr{A} \to \mathbb{R}$ some function.

Let $\mathbf{P}_\lambda := \text{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^3$.

Each face of P_λ has an outward pointing normal vector, and its **upper facets** are those whose normal has positive last coordinate.

The **regular polyhedral subdivision** $\mathscr{T}_\lambda$ of $\Delta_{\mathscr{A}}$ induced by λ is given by the projection of the upper facets back to $\mathbb{R}^2$.

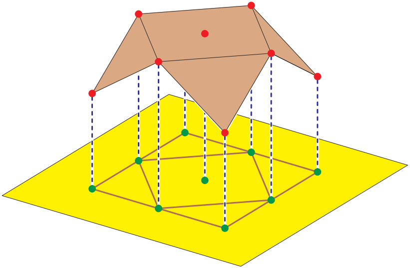


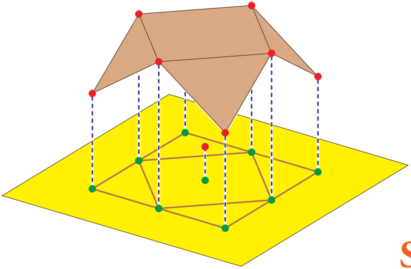


A **decomposition** $\mathscr{D}$ of the configuration $\mathscr{A}$ of points is a collection $\mathscr{D}$ of subsets of $\mathscr{A}$ called **faces**.

The **convex hulls** of these faces are required to be the faces of a **polyhedral subdivision** $\mathscr{T}(\mathscr{D})$ of $\Delta_{\mathscr{A}}$.

The decomposition $\mathscr{D}$ is **regular** if the polyhedral subdivision $\mathscr{T}(\mathscr{D})$ is regular.





Control Surfaces

Let $\mathscr{A} \subset \mathbb{Z}^2$ be a finite set, $w \in \mathbb{R}^{\mathscr{A}}_{>}$ be weights and $\mathscr{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathscr{A}\}$ be control points for a toric patch $Y_{\mathscr{A},w,\mathscr{B}}$ of shape $\mathscr{A}$.

Let $\mathscr{D}$ be a **decomposition** of $\mathscr{A}$. The **control surface** induced by $\mathscr{D}$ is the union

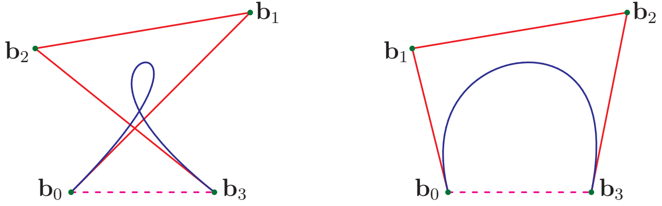
$$\mathbf{Y}_{\mathscr{A},\mathbf{w},\mathscr{B}}(\mathscr{D}) := \bigcup_{\mathscr{F}\in\mathscr{D}} Y_{\mathscr{F},w|_{\mathscr{F}},\mathscr{B}|_{\mathscr{F}}},$$

Faces of toric patches are again toric patches.

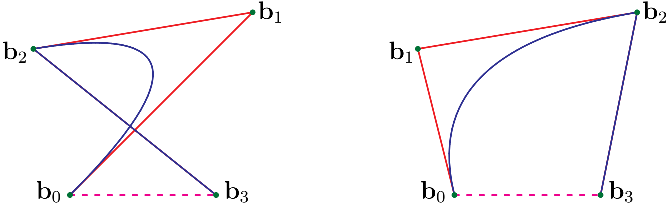
The control surface $Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D})$ is a C^0 spline surface.

$Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D})$ is **regular** if the decomposition $\mathscr{D}$ is regular.

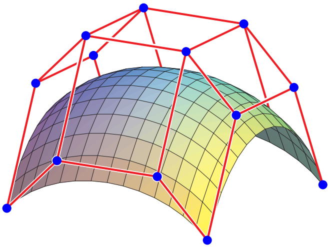
Given the regular decomposition $\{0,1,2\}$, $\{2,3\}$ of $\{0,1,2,3\}$ and the following two rational cubic Bézier planar curves.

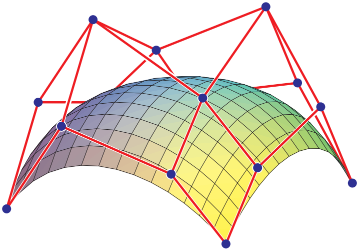


The regular control curves are



Regular Control Surfaces for Rational Bicubic Patches





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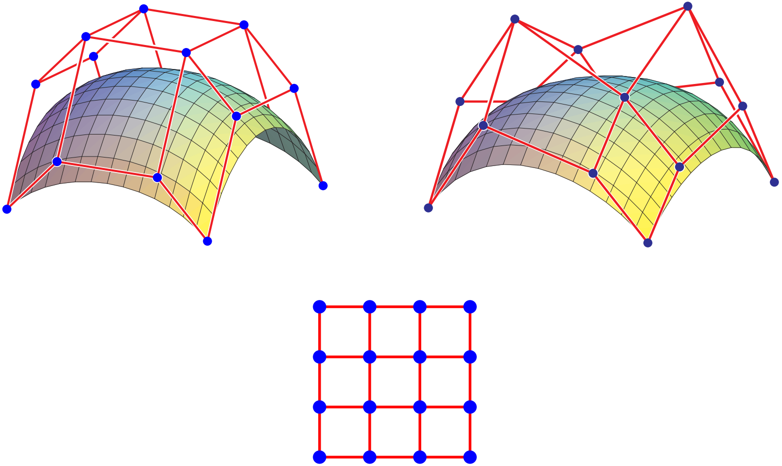

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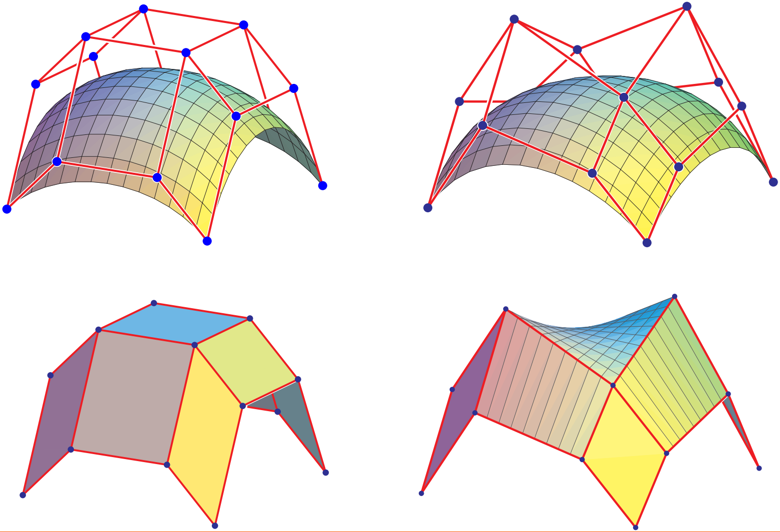


Let $\lambda\colon \mathscr{A} \to \mathbb{R}$ be a **lifting function** and $w = \{w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathscr{A}\}$ a set of weights.

Define $\mathbf{w}_\lambda(\mathbf{t}) := \{t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\}$.

These weights are used to define a **toric degeneration** of the patch,

$$F_{\mathscr{A},w,\mathscr{B},\lambda}(\mathbf{x};t) := \frac{\sum_{\mathbf{a}\in\mathscr{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \beta_{\mathbf{a}}(\mathbf{x})\, \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathscr{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \beta_{\mathbf{a}}(\mathbf{x})}.$$

Let $\mathscr{D}_\lambda$ be the **regular decomposition** of $\mathscr{A}$ induced by λ.

THEOREM

Every regular control surface is the limit of the corresponding patch under a toric degeneration.

$$\lim_{t\to\infty} Y_{\mathscr{A},w,\mathscr{B},\lambda}(t) = Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D}_\lambda).$$

First Main Theorem

THEOREM

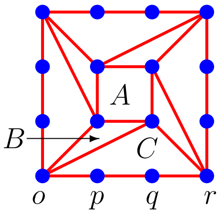
Let $\mathscr{A} \subset \mathbb{Z}^m$ be a finite set and $\mathscr{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^n$ a set of control points. If $Y \subset \mathbb{R}^n$ is a set for which there is a sequence $w^1, w^2, \ldots$ of weights so that

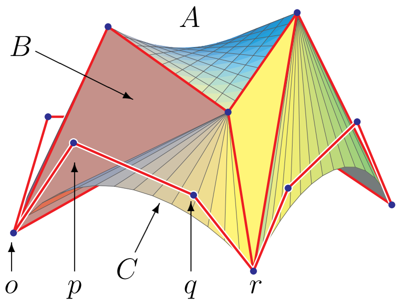
$$\lim_{i\to\infty} Y_{\mathscr{A},w^i,\mathscr{B}} \;=\; Y\,.$$

then there is a lifting function $\lambda\colon \mathscr{A} \to \mathbb{R}$ and a weight $w \in \mathbb{R}^{\mathscr{A}}_{>}$ such that $Y = Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D}_\lambda)$, a regular control surface.

Regular control surfaces are exactly the possible limits of toric patches when the control points $\mathscr{B}$ are fixed but the weights w are allowed to vary.

The irregular control surface below cannot be the limit of toric Bézier patches.





Bibliography

- Gheorghe Craciun, Luis García-Puente, and Frank Sottile, Some geometrical aspects of control points for toric patches, Mathematical Methods for Curves and Surfaces, Lecture Notes in Computer Science, vol. 5862, Springer, 2010, pp. 111–135.
- Luis Garcia-Puente, Frank Sottile, and Chungang Zhu, Toric degenerations of Bézier patches, ACM Transactions on Graphics, Vol. 30, No. 5, Article 110, October 2011.