Linear Precision for Toric Patches

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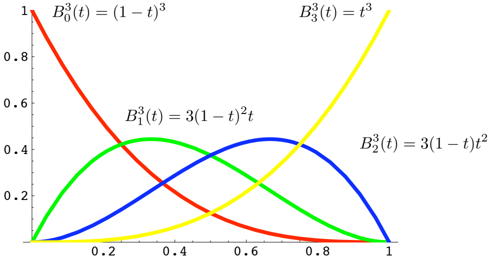
Bézier curves

Bernstein polynomials

$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$



Parametric definition

$$C(t) = \sum_{i=0}^{n} p_i B_i^n(t), \quad t \in [0,1]$$

where $p_0, p_1, \ldots, p_n$ are control points in some affine space.

Properties of Bézier curves

Affine invariance

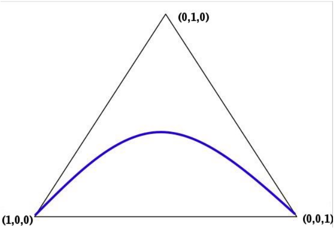
Let $T : \mathbb{A}^m \longrightarrow \mathbb{A}^m$ be an Affine map. Then

$$T(C(t)) = T(\sum_{i=0}^{n} p_i B_i^n(t)) = \sum_{i=0}^{n} T(p_i) B_i^n(t)$$

Convex hull property

The curve $C([0,1])$ is contained in the convex hull of the control points

Endpoint interpolation



Properties of Bézier curves

More properties

- Symmetry
- Pseudo-local control
- Subdivision
- Recursive evaluation

Linear precision

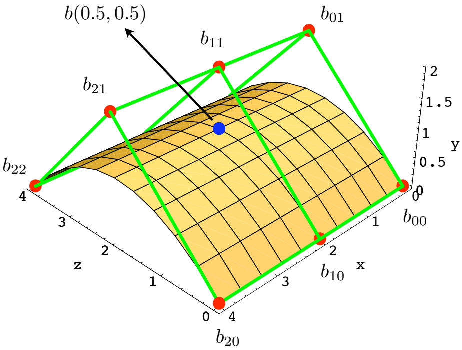
$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$$

$$0(1-t)^3 + t(1-t)^2 + 2t^2(1-t) + t^3 = t$$

Rectangular Bézier surfaces

Parametric representation

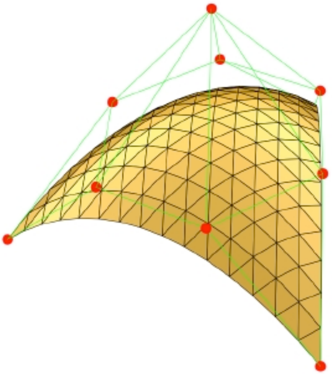
$$\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(s)B_j^n(t)p_{ij}, \quad 0 \leq s, t \leq 1$$



Triangular Bézier surfaces

Parametric representation

$$\sum_{i+j+k=n} \frac{n!}{i!j!k!} u^i v^j w^k p_{ijk}, \quad 0 \leq u, v, w \leq 1 \text{ and } u + v + w = 1$$



Patches

Data

- $\mathcal{A} \subset \mathbb{Z}^d$ finite subset
- $\Delta = \text{conv}(\mathcal{A})$ polytope of dimension d in $\mathbb{R}^d$

Patch

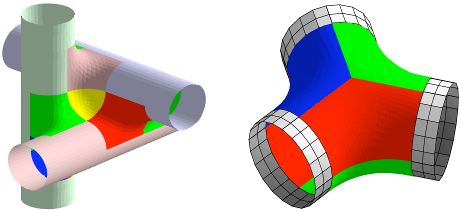
A patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ is a collection of non-negative (blending or basis) functions indexed by $\mathcal{A}$ with common domain Δ and no base points in their domain.

Parametric patch

A set $\{\mathbf{b}_{\mathbf{a}} \in \mathbb{R}^\ell \mid \mathbf{a} \in \mathcal{A}\}$ of control points indexed by $\mathcal{A}$ gives a parametric map

$$\frac{\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x)}$$

Multi-sided patches

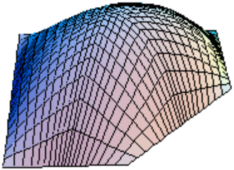


Toric patches

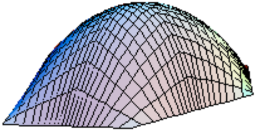
Generalization of Bézier patches

Based on toric varieties

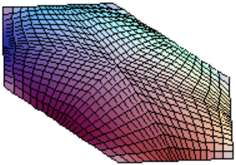
Depend on a polytope and some weights



top view



side view



Toric Bézier patches

- Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon.
- Edges of Δ define lines $h_i(\mathbf{t}) = \langle \mathbf{n}_i, \mathbf{t} \rangle + c_i = 0$, with inward oriented normal primitive lattice vectors $\mathbf{n}_i$.

- Let $\hat{\Delta} = \Delta \cap \mathbb{Z}^2$ be the set of lattice points of Δ
- Note $h_i(\mathbf{a})$ is a non-negative integer for all $\mathbf{a} \in \hat{\Delta}$.

A toric patch associated to Δ is a rational patch with domain Δ and basis functions

$$\beta(s,t) = h_1^{h_1(\mathbf{a})} h_2^{h_2(\mathbf{a})} \cdots h_r^{h_r(\mathbf{a})}.$$

Properties toric Bézier patches

- Affine invariance
- Convex hull
- Boundaries are rational Bézier curves determined by boundary control points
- surfaces interpolate corner control points

Rimvydas Krasauskas

Which toric Bézier patches have linear precision?

Toric varieties

Monomials in d indeterminates

$$x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}, \quad (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$$

Data

- $\mathcal{A} \subset \mathbb{Z}^d$ finite subset
- $\Delta = \text{conv}(\mathcal{A})$ polytope of dimension d in $\mathbb{R}^d$
- $w = \{w_{\mathbf{a}} \in \mathbb{R}_> \mid \mathbf{a} \in \mathcal{A}\}$ set of positive weights indexed by $\mathcal{A}$

Monomial map $\varphi_{\mathcal{A},w} : (\mathbb{C}^*)^d \longrightarrow \mathbb{P}^{\mathcal{A}}$

$$\varphi_{\mathcal{A},w} : \ x \longrightarrow [w_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$$

Translated toric variety

$Y_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}((\mathbb{C}^*)^d)$

Toric patches

Non-negative part of a toric variety

$X_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$

Definition

A Toric patch of shape $(\mathcal{A}, w)$ is any patch β such that the closure X_β of the image of

$$\beta : \Delta \longrightarrow \mathbb{RP}^{\mathcal{A}}, \quad x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}]$$

equals $X_{\mathcal{A},w}$.

Linear precision

Definition

A patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has linear precision if the tautological map τ

$$\tau := \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{a}$$

is the identity function on Δ.

Theorem

A toric patch has a unique reparametrization which has linear precision.

Definition

A toric patch has rational linear precision if its reparametrization having linear precision has blending functions that are rational functions.

Quadratic (rescaled) Bézier curve

Let $\mathcal{A} = \{0, 1, 2\}$ and $w = (1, 2, 1)$, then $X_{[0,2],w}$ is the image of

$$t \mapsto [1, 2t, t^2], \quad t > 0.$$

Let $\beta : [0, 2] \to X_{[0,2],w}$ be given by

$$t \longmapsto [(2-t)^2, 2t(2-t), t^2], \quad t \in [0, 2].$$

The tautological map $\tau : [0, 2] \to [0, 2]$ is given by

$$\frac{0 \cdot (2-t)^2 + 1 \cdot 2t(2-t) + 2t^2}{(2-t)^2 + 2t(2-t) + t^2} = \frac{4t}{4} = t\,.$$

Logarithmic Toric Differential

Laurent polynomial

Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite subset and $w \in \mathbb{R}^{\mathcal{A}}_{>}$ be a system of weights, the the Laurent polynomial $f = f_{\mathcal{A},w}$ is defined by

$$f = f_{\mathcal{A},w} := \sum_{\mathbf{a} \in \mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}}$$

Theorem

A toric patch of shape $(\mathcal{A}, w)$ has rational linear precision if and only if the rational function $\psi_{\mathcal{A},w} : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ defined by

$$D_{\text{torus}} \log f = \frac{1}{f}\left(x_1 \frac{\partial}{\partial x_1} f, x_2 \frac{\partial}{\partial x_2} f, \ldots, x_d \frac{\partial}{\partial x_d} f \right)$$

is a birational isomorphism.

Tautological projection

Definition

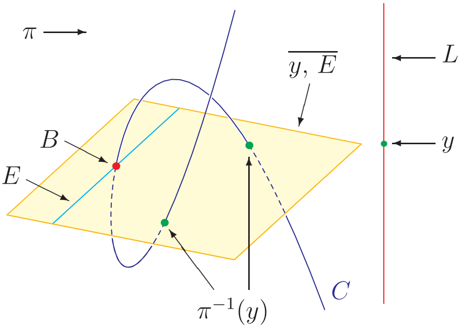
Let $\mathcal{A} \subset \mathbb{R}^d$. Given a point $y = [y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \in \mathbb{RP}^{\mathcal{A}}$, if the sum

$$\sum_{\mathbf{a} \in \mathcal{A}} y_{\mathbf{a}} \cdot (1, \mathbf{a}) \in \mathbb{R}^{d+1}$$

is non-zero then it represents a point in $\mathbb{RP}^d$.

This map is the tautological projection

$$\pi : \mathbb{RP}^{\mathcal{A}} - - \to \mathbb{RP}^d.$$



Geometry of Linear Precision

Universal map given by $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$

$\beta\colon \Delta \to \mathbb{RP}^{\mathcal{A}}, \qquad \beta : x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}]$

Let $X_\beta = \beta(\Delta)$, and $Y_\beta = \overline{X_\beta}$.

Theorem

If a patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has linear precision, then

1. *Y_β is a rational variety,*
2. *almost all codimension d planes L containing the center $E_{\mathcal{A}}$ of the tautological projection meet Y_β in at most one point outside of $E_{\mathcal{A}}$.*

Theorem

The blending functions for the toric patch X_β which have linear precision are given by the coordinates of the inverse of $\pi : X_\beta \longrightarrow \Delta$.

Algebraic Statistics

$X_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$

Algebraic statistics

In statistics $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$ is known as as a log-linear model or discrete exponential family.

Theorem (Darroch and Ratcliff)

The inverse image of the tautological projection can be numerically obtained by the method know as iterative proportional fitting.

Theorem

A toric patch has rational linear precision if and only if the toric model $X_{\mathcal{A},w}$ has maximum likelihood degree 1.