What is Computational Algebraic Geometry?

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Affine Varieties

Let k be a field. An affine variety is the common zero locus of polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$.

$$V(f_1, \ldots, f_r) = \{f_1 = 0, f_2 = 0, \ldots, f_r = 0\}$$

Trivial Examples

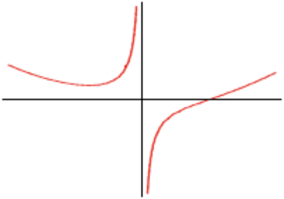
- $V(1) = \emptyset$
- $V(0) = k^n$

Linear Varieties

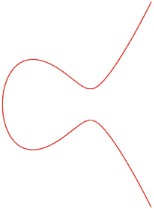
For linear polynomials f_i, $V(f_1, \ldots, f_r)$ is the solution space of an inhomogeneous system of linear equations. This variety is described parametrically applying Gauss Algorithm.

Planar Curves

The graph of the function $y = \dfrac{x^3 - 1}{x}$ is the variety $V(xy - x^3 + 1)$.

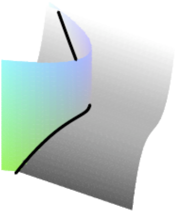


Let $f(x, y) = y^2 - x^3 - x^2 + 2x - 1$, then $V(f)$ is the plane curve:

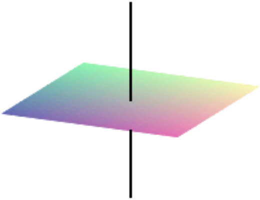


More Examples

The twisted cubic $V(y - x^2, z - x^3)$



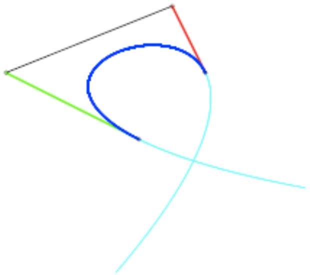
$V(xz, yz) = V(z) \cup V(x, y)$



Parametric Varieties

The Bézier curve $C \subset k^2$ parametrized by

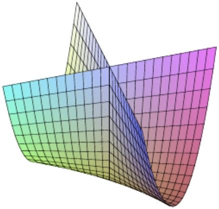
$$X(t) = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$Y(t) = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$



Parametric Varieties

Whitney umbrella given by the parametrization

$$X(s,t) = st, \qquad Y(s,t) = s \qquad Z(s,t) = t^2$$



We need to describe these geometric objects by implicit equations, i.e., as $V(f_1, \ldots, f_r)$. This allows to easily check if a given point lies on the variety.

Ideals

Given $S \subset k^n$ form

$$I(S) = \{g \in k[x_1, \ldots, x_n] \mid g(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in S\}$$

If f and $g \in I(S)$ and h arbitrary then

$$f + g \in I(S) \text{ and } hf \in I(S).$$

So $I(S)$ is an ideal: the ideal of polynomial functions vanishing on S.

Hilbert Basis Theorem

Ideals

Given $f_1, \ldots, f_r$ in $k[x_1, \ldots, x_n]$,

$$\langle f_1, \ldots, f_r \rangle = \{\sum_{i=i}^{n} h_i f_i \mid h_1, \ldots, h_n \in k[x_1, \ldots, x_n]\}$$

is the ideal generated by $f_1, \ldots, f_r$.

Theorem

If $I \subset k[x_1, \ldots, x_n]$ is an ideal, there exists $f_1, \ldots, f_r$ such that

$$\langle f_1, \ldots, f_r \rangle = I.$$

$I(S)$ has a finite generating set.

Rational Parametrization

Let $p_1(t_1, \ldots, t_d), \ldots, p_n(t_1, \ldots, t_d) \in k[t_1, \ldots, t_d]$

$$S = \{(p_1(a_1, \ldots, a_d), \ldots, p_n(a_1, \ldots, a_d)) \in k^n \mid (a_1, \ldots, a_d) \in k^d\}$$

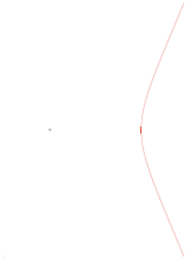
is called a rational parametrization (r.p.).

Theorem

If S is a r.p. and $I(S) = \langle f_1, \ldots, f_r \rangle$ then S and $V(f_1, \ldots, f_r)$ differ by a set of dimension less than the dimension of S.

Example

Let $S = \{(t^2 + 1, t^3 + t) \mid t \in \mathbb{R}\}$. Then $I(S) = \langle y^2 - x^3 - x^2 \rangle$



Quadratic Bézier Curves

Example: Hardy-Weinberg Equilibrium

Consider a gene with two alleles: A and B. Let X = # of times A appears in a pair of matching chromosomes.

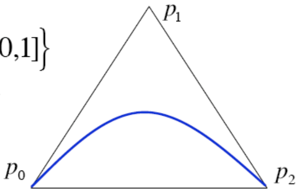
If A and B are in Hardy-Weinberg equilibrium, the alleles are selected independently on each chromosome, that is, there exists θ such that $P(X=0)=\theta^2$

$P(X=1)=2\theta(1-\theta)$ and $P(X=2)=(1-\theta)^2$

$$S=\{(\theta^2, 2\theta(1-\theta), (1-\theta)^2) \mid \theta \in [0,1]\}$$

$$I(S)=\langle p_0+p_1+p_2-1,\ p_1^2-4p_0p_2\rangle$$

$$\subset \mathbf{C}[p_0, p_1, p_2]$$



Affine Varieties

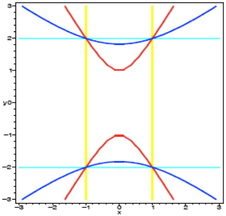
Varieties

For an ideal $I \subset k[x_1, \ldots, x_n]$,

$$V(I) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I\}$$

$V(f_1, \ldots, f_r) = V(\langle f_1, \ldots, f_r \rangle)$

$$\left\langle 2x^2 - 3y^2 + 10, 3x^2 - y^2 + 1 \right\rangle = \left\langle x^2 - 1, y^2 - 4 \right\rangle$$



Projection and Elimination

In linear algebra, Gauss Algorithm parametrizes the solution space of a linear system of equations by a coordinate space. This can be viewed as a projection of the solution space.

Projection is also applied to solve systems of polynomial equations

For any ideal $I \subset k[x_1, \ldots, x_n]$ we consider the elimination ideal

$$I_m = I \cap k[x_{m+1}, \ldots, x_n]$$

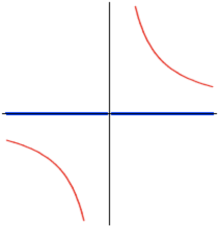
and the projection $\pi_m : k^n \longrightarrow k^{n-m}$ given by

$$\pi_m(a_1, \ldots, a_n) = (a_{m+1}, \ldots, a_n)$$

Example

Let $I = \langle x^2 - 1, y^2 - 4 \rangle$, then $V(I) = \{(1,2), (-1,2), (1,-2), (-1,-2)\}$ then $\pi_1(V(I)) = \{-2, 2\}$ and $I_1 = \langle y^2 - 4 \rangle$.

Let $S = V(xy - 1)$, then $\pi_1(S) = k \setminus \{0\}$.



Theorem

If k is algebraically closed, then $\overline{\pi_m(V(I))} = V(I_m)$

Gröbner bases

Theorem

If $G = \{g_1, \ldots, g_r\}$ is a Gröbner basis of $I \subset k[x_1, \ldots, x_n]$ with respect to the lexicographic ordering, then

$$G_m = G \cap k[x_{m+1}, \ldots, x_n]$$

is a Gröbner basis of $I_m \subset k[x_{m+1}, \ldots, x_n]$ with respect to the lexicographic ordering.

Polynomial Implicitization

Let $S = \{(t^2, t^3, t^4) \mid t \in k\}$. Consider the ideal

$$I = \left\langle x - t^2, y - t^3, z - t^4 \right\rangle$$

A Gröbner basis of I with respect to the lexicographic order with $t > z > y > x$ is given by $\{y^2 - x^3, z - x^2, tx - y, ty - x^2, t^2 - x\}$, hence

$$\overline{\pi_1(V(I))} = V(I_1) = V(y^2 - x^3, z - x^2)$$

Rational Implicitization

Theorem

Suppose k is an infinite field and we are given a rational map $\phi : k^m \setminus Z \longrightarrow k^n$ given by

$$(t_1, \ldots, t_m) \longmapsto \left(\frac{f_1(t)}{g_1(t)}, \ldots, \frac{f_n(t)}{g_n(t)}\right)$$

with f_i and $g_i \in k[t_1, \ldots, t_m]$ and $Z = V(g)$, $g = g_1 \cdot g_2 \cdots g_n$. Then

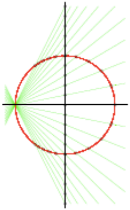
$$\overline{\mathrm{Im}(\phi)} = V(I_{m+1})$$

with

$$I = \langle g_1 x_1 - f_1, \ldots, g_n x_n - f_n, 1 - gs \rangle \subset k[s, t_1, \ldots, t_m, x_1, \ldots, x_n]$$

Example

Consider the parametrization of the circle given by
$t \longmapsto \left(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}\right)$
given by the 2^{nd} intersection point of $y = t(x+1)$ with the circle:



A Gröbner basis of
$I = \langle (t^2+1)x - (1-t^2), (t^2+1)y - 2t, 1 - (t^2+1)^2 s \rangle$ with respect to lex with $s > t > x > y$ is given by

$$\{x^2 + y^2 - 1, ty + x - 1, tx + t - y, s - \frac{1}{2}x - \frac{1}{2}\}$$

and hence $I_2 = \langle x^2 + y^2 - 1 \rangle$.